

# On the Mutual Information Distribution of OFDM-Based Spatial Multiplexing: Exact Variance and Outage Approximation

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## Abstract

This paper considers the distribution of the mutual information of frequency-selective spatially-uncorrelated Rayleigh fading MIMO channels. Results are presented for OFDM-based spatial multiplexing. New exact closed-form expressions are derived for the variance of the mutual information. In contrast to previous results, our new expressions apply for systems with both arbitrary numbers of antennas and arbitrary-length channels. Simplified expressions are also presented for high and low SNR regimes. The analytical variance results are used to provide accurate analytical approximations for the distribution of the mutual information, and the outage capacity.

## Index Terms

MIMO Systems, Orthogonal Frequency Division Multiplexing, Mutual Information

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## I. INTRODUCTION

Multiple-input multiple-output (MIMO) antenna technology has emerged as an effective technique for significantly improving the capacity of wireless communication systems. A great deal of work has been done on analyzing the MIMO capacity in various flat-fading channel scenarios, since the pioneering work of [1] and [2]. In particular, the mean (ergodic) capacity has now been comprehensively investigated (e.g. see [3–18] and references therein).

In addition, the outage capacity has also been investigated for flat-fading channels. This is an important capacity measure for systems with stringent delay constraints, and also provides information about the system diversity [19]. With the exception of the exact two/three antenna results presented in [20, 21], outage capacity analysis has typically involved approximating the distribution of the mutual information, since exact closed-form solutions are not forthcoming. It has been shown that the Gaussian distribution provides a good approximation in many cases [5, 8, 13, 22, 23].

In this paper, we consider frequency-selective MIMO channels, which are applicable for many current high data-rate wireless systems. We focus on MIMO orthogonal frequency-division multiplexing (OFDM) systems, since they form the underlying technology for a many emerging MIMO standards, as summarized in Table I, and consider spatial multiplexing transmission. Despite their key practical significance however, for these systems (and indeed frequency-selective MIMO channels in general) there are relatively few analytic MIMO capacity results. The ergodic capacity (average mutual information) was considered in [19, 24–26] and [27, 28], assuming Rayleigh and Rician channels respectively, and was found to be easily obtained by summing the equivalent flat-fading ergodic MIMO capacity of each individual OFDM subcarrier. In contrast, the outage capacity does not decompose in this way.

Calculating the outage capacity for frequency-selective channels is difficult due to the non-negligible correlations between subcarrier channel matrices. As such, the investigation of outage

capacity has usually been performed using simulation studies [19, 29, 30]. It appears that the only current analytical outage capacity results for frequency-selective channels are presented in [31], [32] and [33], all of which derive a Gaussian approximation for the mutual information distribution. The results in [31] however, are based on deriving exact expressions for the mutual information variance of single-input single-output (SISO) channels only; whereas the results in [32] and [33] are based on approximating the mutual information variance using asymptotic methods. Specifically, [32] considers multiple-input single-output (MISO) channels with asymptotically large channel lengths, whereas [33] considers MIMO channels with infinite numbers of transmit and receive antennas. We note also that for the extreme frequency-selective fading case, ie. where the MIMO subcarrier matrices are independent across frequency, the variance of the mutual information could be easily calculated by adapting known MIMO flat-fading variance results given, for example, in [8] and [15]. For many practical systems however, the subcarrier channels are typically highly correlated across frequency, and this approach cannot be applied.

In this paper, we consider MIMO OFDM-based spatial multiplexing systems with finite numbers of antennas, and operating over spatially-uncorrelated Rayleigh fading channels with finite delay spreads. We first derive new exact closed-form expressions for the mutual information variance. We also give explicit reduced formulas for the specific cases of multiple-input single-output (MISO), single-input multiple-output (SIMO), and single-input single-output (SISO) systems. Moreover, simplified closed-form expressions are derived for the variance in the high and low signal-to-noise ratio (SNR) regimes.

Based on the new analytic variance results (along with known analytic mean results), we then present new approximations to the mutual information distribution of OFDM-based spatial multiplexing systems. In particular, we present a new closed-form Gaussian approximation, which is shown to be extremely accurate for many different system and channel scenarios. In the low SNR regime, we also present a new analytic Gamma approximation, which we show to be more accurate than the Gaussian approximation in this case.

Finally, we use the analytic Gaussian approximation to estimate the outage capacity. We find that the approximation is very accurate, and show that for outage levels of practical interest, the outage capacity depends heavily on the delay spread of the channel.

The paper is organized as follows. In Section II we describe the frequency-selective MIMO channel model, the OFDM-based spatial multiplexing signal model, and the associated mutual information. In Section III, we present the main analytical contributions of the paper, namely, analytical expressions for the variance of the mutual information. The proofs are relegated to the appendices. In Section IV, we approximate the distribution of the mutual information, and investigate the outage capacity.

The following notation is used throughout this paper. Matrices are represented with uppercase boldface, and vectors with lowercase boldface. The superscripts  $(\cdot)^T$ ,  $(\cdot)^*$ , and  $(\cdot)^\dagger$  indicate matrix transpose, complex conjugate, and complex conjugate transpose respectively. The matrix  $\mathbf{I}_p$  denotes a  $p \times p$  identity matrix. We use  $\det(\cdot)$  and  $\text{tr}(\cdot)$  to represent the matrix determinant and trace operations respectively. The operator  $E[\cdot]$  denotes expectation, and  $\text{Var}(\cdot)$  denotes variance. The real Gaussian distribution with zero-mean and unit-variance is denoted  $\mathcal{N}(0, 1)$ , the corresponding complex circularly symmetric Gaussian distribution is denoted  $\mathcal{CN}(0, 1)$ , and the chi-square distribution with  $r$  degrees of freedom is denoted  $\chi_r^2$ .

## II. OFDM-BASED SPATIAL MULTIPLEXING SYSTEMS

### A. Channel and Signal Model

We consider a single-user OFDM-based spatial multiplexing system employing  $N_t$  transmit antennas,  $N_r$  receive antennas, and  $N$  subcarriers. The channel is assumed to be frequency-selective and is modeled as a length- $L$  finite impulse-response (FIR) filter (as in [19, 25]), for which the discrete-time input-output relation is given by [25]

$$\mathbf{y}[q] = \sum_{p=0}^{L-1} \sigma_p \mathbf{H}[p] \mathbf{x}[q-p] + \mathbf{n}[q] \quad (1)$$

where  $\mathbf{x}[q] \in \mathcal{C}^{N_t \times 1}$  is the signal vector transmitted at sample index  $q$ ,  $\mathbf{y}[q] \in \mathcal{C}^{N_r \times 1}$  is the corresponding received signal vector, and  $\mathbf{n}[q] \in \mathcal{C}^{N_r \times 1}$  is the noise vector containing independent elements  $\sim \mathcal{CN}(0, 1)$ . Also,  $\sigma_p$ , for  $p = 0, \dots, L-1$ , represents the channel power delay profile, and is normalized according to

$$\sum_{p=0}^{L-1} \sigma_p^2 = 1. \quad (2)$$

The  $N_r \times N_t$  random matrices  $\mathbf{H}[p]$ , for  $p = 0, \dots, L-1$ , represent the MIMO channel impulse response. These matrices are assumed to be mutually uncorrelated, and are assumed to be known perfectly at the receiver but are unknown at the transmitter. The channel is assumed to be quasi-static, remaining constant for the duration of a codeword, but changing independently from codeword to codeword. Throughout the paper, we assume that the channel elements exhibit spatially-uncorrelated Rayleigh fading<sup>1</sup>, in which case each  $\mathbf{H}[p]$  contains independent elements  $\sim \mathcal{CN}(0, 1)$ .

At the transmitter, the time-domain input sequence  $\mathbf{x}[q]$  is generated as  $N_t$  parallel OFDM symbols. The symbols for each antenna are OFDM modulated using an  $N$ -point inverse fast-Fourier transform (IFFT) prior to transmission. At the receiver, OFDM demodulation is performed at each receive antenna using an  $N$ -point FFT. A key advantage of OFDM-based spatial multiplexing is that equalization is simple, since the frequency-selective MIMO channel is transformed into  $N$  orthogonal flat-fading MIMO subchannels via the IFFT/FFT processing.

To maintain orthogonality in the presence of intersymbol interference caused by multipath, OFDM systems typically employ a cyclic prefix extension. Assuming that the cyclic prefix is longer than the delay spread of the channel, we can write the equivalent frequency domain

<sup>1</sup>Note that a number of recent investigations have studied the impact of spatial correlation on MIMO capacity (see, eg. [8, 9, 12, 13]). We do not follow this line of work here however, since our primary focus is to study the impact of frequency-selective fading on capacity, in which case the effect of correlation is observed across frequency.

input-output model for OFDM-based spatial multiplexing as follows

$$\mathbf{r}_k = \mathbf{H}_k \mathbf{a}_k + \mathbf{n}_k, \quad k = 0, \dots, N-1 \quad (3)$$

where  $\mathbf{a}_k$  is the transmitted vector for the  $k$ th subcarrier, assumed to be i.i.d. Gaussian with covariance matrix  $E[\mathbf{a}_k \mathbf{a}_k^\dagger] = \frac{\gamma}{N_t} \mathbf{I}_{N_t}$ ,  $\mathbf{r}_k$  is the received vector for the  $k$ th subcarrier, and  $\mathbf{n}_k$  is the corresponding complex AWGN vector satisfying  $E[\mathbf{n}_k \mathbf{n}_\ell^\dagger] = \mathbf{I}_{N_r} \delta[k - \ell]$ , where  $\delta[\cdot]$  is the Kronecker-delta function. Also,  $\mathbf{H}_k$  is the  $k$ th subcarrier channel matrix given by

$$\mathbf{H}_k = \sum_{p=0}^{L-1} \sigma_p \mathbf{H}[p] \exp\left(-j2\pi \frac{k}{N} p\right) \quad (4)$$

containing independent entries  $(\mathbf{H}_k)_{i,j} \sim \mathcal{CN}(0, 1)$ . Note that due to the finite-length impulse response, correlation exists between different subcarrier channel matrices. Using (4), the correlation coefficients between the channel elements on two arbitrary subcarriers  $k$  and  $\ell$  is easily derived as follows (see also [34])

$$\rho_{k-\ell} = E[(\mathbf{H}_k)_{i,j} (\mathbf{H}_\ell)_{i',j'}^*] = \sum_{p=0}^{L-1} \sigma_p^2 e^{-j2\pi(k-\ell)p/N} \delta[i - i'] \delta[j - j'] \quad (5)$$

for all  $i, j, i', j'$ . As expected, these frequency correlation coefficients depend only on the difference between subcarriers (i.e.  $k - \ell$ ), and not on the subcarriers themselves.

Note that with the above model, the SNR per receive antenna per subcarrier (henceforth referred to as ‘the SNR’) is given by  $\gamma$ .

### B. Mutual Information

The focus of this paper is on the statistics of the mutual information of OFDM-based spatial multiplexing systems. It is now well-known that the instantaneous mutual information in b/s/Hz for a given channel realization is given by [19]

$$\mathcal{I}_{\text{ofdm}} = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{I}_k \quad (6)$$

where  $\mathcal{I}_k$  is the instantaneous mutual information for the  $k$ th OFDM subcarrier, given by

$$\mathcal{I}_k = \log_2 \det \left( \mathbf{I}_{N_r} + \frac{\gamma}{N_t} \mathbf{H}_k \mathbf{H}_k^\dagger \right). \quad (7)$$

Note that the loss in mutual information due to the cyclic prefix has been neglected in (7). The mean (ergodic) mutual information is given by

$$E[\mathcal{I}_{\text{ofdm}}] = \frac{1}{N} \sum_{k=0}^{N-1} E[\mathcal{I}_k]. \quad (8)$$

It is obvious that (8) is equivalent to the ergodic mutual information of a flat-faded channel, for which case closed-form expressions are now available [6, 15].

### III. VARIANCE OF THE MUTUAL INFORMATION

In this section we derive new closed-form expressions for the variance of the mutual information of OFDM-based spatial multiplexing. Our results are exact, and apply for arbitrary finite system and channel parameters. We also present simplified expressions for the variance in the high and low SNR regimes, and give explicit reduced variance expressions for the cases of MISO, SIMO, and SISO systems. These results will be subsequently used in Section IV for providing accurate approximations to the mutual information distribution, and to the outage capacity.

#### A. Exact Analysis at All SNRs

The following theorem presents an exact expression for the variance of the mutual information of MIMO-OFDM systems.

*Theorem 1:* The variance of the mutual information of MIMO-OFDM systems is given by

$$\text{Var}(\mathcal{I}_{\text{ofdm}}) = \frac{(\log_2(e))^2}{\Gamma_m(n)\Gamma_m(m)} \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d)\varphi(\rho_d) + \sum_{r=1}^m \sum_{s=1}^m \frac{\det(\mathbf{B}_{r,s})}{N} - \frac{(\sum_{r=1}^m \det(\mathbf{A}_r))^2}{\Gamma_m(n)\Gamma_m(m)} \right) \quad (9)$$

where  $m = \min(N_r, N_t)$ ,  $n = \max(N_r, N_t)$ ,  $\Gamma_m(\cdot)$  is the complex multivariate gamma function

defined as

$$\Gamma_m(n) = \prod_{i=1}^m \Gamma(n - i + 1) \quad (10)$$

and

$$\varphi(\rho_d) = \begin{cases} \frac{(\sum_{r=1}^m \det(\mathbf{A}_r))^2}{\Gamma_m(n)\Gamma_m(m)} & \text{for } |\rho_d| = 0 \\ \sum_{r=1}^m \sum_{s=1}^m e^{2N_t/\gamma} \det(\mathbf{C}_{r,s}(\rho_d)) & \text{for } 0 < |\rho_d| < 1 \\ \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{B}_{r,s}) & \text{for } |\rho_d| = 1 \end{cases} \quad (11)$$

The matrices  $\mathbf{A}_r$ ,  $\mathbf{B}_{r,s}$  and  $\mathbf{C}_{r,s}(\cdot)$  are  $m \times m$ , with  $(i, j)^{\text{th}}$  elements

$$(\mathbf{A}_r)_{i,j} = \begin{cases} b! & \text{for } j \neq r \\ b!e^{N_t/\gamma} g_1(b+1) & \text{for } j = r \end{cases}, \quad (12)$$

$$(\mathbf{B}_{r,s})_{i,j} = \begin{cases} b! & \text{for } j \neq r \text{ and } j \neq s \\ b!e^{N_t/\gamma} g_1(b+1) & \text{for } j = r \text{ or } j = s, \text{ and } r \neq s \\ 2(N_t/\gamma)^{b+1} e^{N_t/\gamma} \sum_{t=0}^b \binom{b}{t} (-1)^t \\ \quad \times G_{3,4}^{4,0} \left( N_t/\gamma \middle|_{0,t-b-1,t-b-1,t-b-1}^{t-b,t-b,t-b} \right) & \text{for } j = r = s \end{cases} \quad (13)$$

and

$$(\mathbf{C}_{r,s}(\rho_d))_{i,j} = \begin{cases} \eta_{i,j}(1, \rho_d) & \text{for } i \neq r, j \neq s \\ \eta_{i,j}(g_1(z), \rho_d) & \text{for } i = r, j \neq s \\ |\rho_d|^{2(i-j)} \eta_{j,i}(g_1(z), \rho_d) & \text{for } i \neq r, j = s \\ \frac{(1-|\rho_d|^2)^z}{|\rho_d|^{2(j-1)}} e^{\frac{2N_t|\rho_d|^2}{\gamma(1-|\rho_d|^2)}} \sum_{t=0}^{\infty} \frac{|\rho_d|^{2t} \Gamma(u)\Gamma(v)g_2(u)g_2(v)}{t!(\tau+t)!} & \text{for } i = r, j = s \end{cases} \quad (14)$$

respectively. Also,  $b = n + m - i - j$ ,  $\tau = n - m$ ,  $z = \tau + i + j - 1$ ,  $u = \tau + i + t$ ,  $v = \tau + j + t$ ,

and  $G_{3,4}^{4,0}(\cdot)$  is the Meijer-G function (see [35, eq. (9.301)] for definition),

$$g_1(z) = \sum_{h=1}^z E_h \left( \frac{N_t}{\gamma} \right) \quad (15)$$

$$g_2(z) = \sum_{h=1}^z E_h \left( \frac{N_t}{\gamma(1-|\rho_d|^2)} \right) \quad (16)$$

where  $E_h(\cdot)$  is the Exponential Integral (see [36, eq. (5.1.12)] for definition). The function  $\eta_{i,j}(\cdot, \cdot)$

is defined as

$$\eta_{i,j}(f(z), \rho_d) = \Gamma(\tau + j) \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{1 - |\rho_d|^2}{|\rho_d|^2} \right)^t (\tau + j - t)_{i-1} f(z - t) \quad (17)$$

for an arbitrary input function  $f$ , and  $(\cdot)_r$  is the Pochhammer symbol

$$(a)_r = a \cdot (a + 1) \cdot \dots \cdot (a + r - 1) = \frac{\Gamma(a + r)}{\Gamma(a)}; \quad (a)_0 = 1. \quad (18)$$

*Proof:* See Appendix I.  $\square$

Note that the exact variance expression in Theorem 1 can be easily evaluated since it primarily involves simple polynomial and exponential terms, as well as standard functions such as exponential integrals and Meijer- $G$  functions, both of which are implemented as built-in procedures in various mathematical software packages such as Maple and Mathematica. We also note that although Theorem 1 involves infinite series of exponential integrals, its numerical evaluation can be made more efficient by exploiting the following recurrence relations [36, eqs. (5.1.7) and (5.1.14)]

$$\begin{aligned} E_1(z) &= -\text{Ei}(-z) \\ E_{n+1}(z) &= \frac{1}{n} (e^{-z} - zE_n(z)) \end{aligned} \quad (19)$$

for  $z > 0$ . As such, only a single exponential integral must be explicitly evaluated when summing these series. Moreover, it turns out that this infinite series converges quickly, and can generally be evaluated with less than 20 terms. Therefore the computational challenge associated with this series is very low.

The following corollary presents an exact variance expression for the mutual information of SIMO and MISO OFDM systems (i.e. cases with  $m = 1, n > 1$ ). To the best of our knowledge, this result is also new.

*Corollary 1:* The variance of the mutual information of SIMO/MISO-OFDM systems is given by

$$\text{Var}(\mathcal{I}_{\text{ofdm}}) = \frac{(\log_2(e))^2}{\Gamma(n)} \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N - d) \varphi(\rho_d) - \Gamma(n) e^{2N_t/\gamma} (g_1(n))^2 \right)$$

$$+ \frac{2e^{N_t/\gamma}}{N} \left(\frac{N_t}{\gamma}\right)^n \sum_{t=0}^{n-1} \binom{n-1}{t} (-1)^{n-1-t} G_{3,4}^{4,0} \left(N_t/\gamma \Big|_{0,-t-1,-t-1,-t-1}^{-t,-t,-t}\right) \quad (20)$$

where

$$\varphi(\rho_d) = \begin{cases} \Gamma(n)e^{2N_t/\gamma}(g_1(n))^2 & \text{for } |\rho_d| = 0 \\ (1 - |\rho_d|^2)^n e^{\frac{2N_t}{\gamma(1-|\rho_d|^2)}} \sum_{t=0}^{\infty} \frac{|\rho_d|^{2t} \Gamma(n+t)(g_2(n+t))^2}{t!} & \text{for } 0 < |\rho_d| < 1 \\ 2e^{N_t/\gamma} \left(\frac{N_t}{\gamma}\right)^n \sum_{t=0}^{n-1} \binom{n-1}{t} (-1)^{n-1-t} G_{3,4}^{4,0} \left(N_t/\gamma \Big|_{0,-t-1,-t-1,-t-1}^{-t,-t,-t}\right) & \text{for } |\rho_d| = 1 \end{cases} \quad (21)$$

□

The following corollary presents an exact variance expression for the mutual information of SISO OFDM systems (i.e.  $m = 1, n = 1$ ).

*Corollary 2:* The variance of the mutual information of SISO-OFDM systems is given by

$$\text{Var}(\mathcal{I}_{\text{ofdm}}) = (\log_2(e))^2 \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d)\varphi(\rho_d) + \frac{2e^{1/\gamma}}{\gamma N} G_{3,4}^{4,0} \left(1/\gamma \Big|_{0,-1,-1,-1}^{0,0,0}\right) - e^{2/\gamma}(g_1(1))^2 \right) \quad (22)$$

where

$$\varphi(\rho_d) = \begin{cases} e^{2/\gamma}(g_1(1))^2 & \text{for } |\rho_d| = 0 \\ (1 - |\rho_d|^2) e^{\frac{2}{\gamma(1-|\rho_d|^2)}} \sum_{t=0}^{\infty} |\rho_d|^{2t} (g_2(1+t))^2 & \text{for } 0 < |\rho_d| < 1 \\ \frac{2e^{1/\gamma}}{\gamma N} G_{3,4}^{4,0} \left(1/\gamma \Big|_{0,-1,-1,-1}^{0,0,0}\right) & \text{for } |\rho_d| = 1 \end{cases} \quad (23)$$

□

Very recently, an equivalent expression for the SISO-OFDM variance given in (22) was presented in [31]<sup>2</sup>. In contrast to (22) however, the equivalent result from [31] is not expressed in closed-form, and it requires the evaluation of infinite series of incomplete gamma functions.

In Fig. 1 we compare the analytical variance expression (9) with the variance obtained via Monte-Carlo simulation. Results are presented for two different  $N_t \times N_r$  antenna configurations as a function of the channel length  $L$ . A uniform power delay profile is assumed (i.e.  $\sigma_p^2 = 1/L$ ,

<sup>2</sup>Note that this expression was not explicitly stated in [31]. It can however be trivially obtained by following the derivation of (47) and using [31, Eqs. (12), (41), and (48)].

for  $p = 0, \dots, L - 1$ , 8 subcarriers are used (simply by way of example; similar results are obtained for higher numbers of subcarriers), and the SNR is set to 10dB. In all cases we see a precise agreement between the simulated and analytic curves. Moreover, the variance is seen to be largest for the system with the least antennas, regardless of the channel length. For both antenna configurations, we see that the variance reduces with increasing  $L$ , and that this reduction is most significant for small  $L$ . For example, by increasing the channel length from  $L = 1$  (flat-fading) to  $L = 2$ , the variance for both antenna configurations is more than halved.

In Fig. 2 we plot the analytical variance expression (9) and Monte-Carlo simulation results for different SNRs, as a function of  $L$ . Again we see a precise agreement between the analytical and simulated results. From this figure we see that for a given channel length the variance of the mutual information varies monotonically with the SNR. This increase is most significant for small values of  $L$ .

### B. Analysis at High SNR

The following theorem presents a closed-form expression for the variance of the mutual information of MIMO-OFDM in the high SNR regime. This result is simpler than the exact general variance result given in Theorem 1, as it does not involve any infinite series.

*Theorem 2:* In the high SNR regime, the variance of the mutual information of MIMO-OFDM systems is given by

$$\text{Var}^\infty(\mathcal{I}_{\text{ofdm}}) = (\log_2(e))^2 \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) \tilde{\varphi}(\rho_d) + \frac{1}{N} \sum_{t=0}^{m-1} \psi'(n-t) - \frac{N-1}{N} \left( \sum_{t=0}^{m-1} \psi(n-t) \right)^2 \right) \quad (24)$$

where

$$\tilde{\varphi}(\rho_d) = \begin{cases} \left( \sum_{t=0}^{m-1} \psi(n-t) \right)^2 & \text{for } |\rho_d| = 0 \\ \frac{\sum_{r=1}^m \sum_{s=1}^m \det(\tilde{\mathbf{C}}_{r,s}(\rho_d))}{\Gamma_m(n) \Gamma_m(m)} & \text{for } 0 < |\rho_d| < 1 \\ \sum_{t=0}^{m-1} \psi'(n-t) + \left( \sum_{t=0}^{m-1} \psi(n-t) \right)^2 & \text{for } |\rho_d| = 1 \end{cases} \quad (25)$$

where  $\tilde{\mathbf{C}}_{r,s}(\rho_d)$  is an  $m \times m$  matrix with  $(i, j)^{\text{th}}$  element for the cases  $i \neq r$  or  $j \neq s$  given by

$$\left(\tilde{\mathbf{C}}_{r,s}(\rho_d)\right)_{i,j} = \begin{cases} \eta_{i,j}(1, \rho_d) & \text{for } i \neq r, j \neq s \\ \eta_{i,j}(\psi(z), \rho_d) & \text{for } i = r, j \neq s \\ |\rho_d|^{2(i-j)} \eta_{j,i}(\psi(z), \rho_d) & \text{for } i \neq r, j = s \end{cases} \quad (26)$$

and for the case  $(i = r, j = s)$  by

$$\begin{aligned} \left(\tilde{\mathbf{C}}_{r,s}(\rho_d)\right)_{i,j} &= h^2(\rho_d) |\rho_d|^{2(i-j)} \eta_{j,i}(1, \rho_d) + |\rho_d|^{2(i'-j)} \eta_{j',i'}(\xi_{|\rho_d|^2}(z-1), \rho_d) + h(\rho_d) (\eta_{i,j}(H(z-1), \rho_d) \\ &\quad - \ln(1 - |\rho_d|^2) \eta_{i,j}(1, \rho_d) + |\rho_d|^{2(i-j)} (\eta_{j,i}(H(z-1), \rho_d) - \ln(1 - |\rho_d|^2) \eta_{j,i}(1, \rho_d))) \end{aligned} \quad (27)$$

where  $i' = \max(i, j)$  and  $j' = \min(i, j)$ . Also,  $\eta_{i,j}(\cdot)$  is defined in (17) in Theorem 1,  $\xi(\cdot)$  is defined in (126),  $h(\cdot)$  is given by

$$h(\rho_d) = \ln(1 - |\rho_d|^2) - \mathcal{K}, \quad (28)$$

and  $\mathcal{K} = 0.5772\dots$  is the Euler-Mascheroni constant. The function  $H(\cdot)$  denotes the harmonic number

$$H(z) = \begin{cases} \sum_{\ell=1}^z \frac{1}{\ell} & \text{for } z > 0 \\ 0 & \text{for } z = 0 \end{cases} \quad (29)$$

and  $\psi(\cdot)$  is the digamma function defined as [36, eq. (6.3.2)]

$$\psi(n-t) = H(n-t-1) - \mathcal{K} \quad (30)$$

with first derivative  $\psi'(\cdot)$  corresponding to the polygamma function [36, eq. (6.4.1)].

*Proof:* See Appendix III. □

The following two corollaries present very simple high SNR variance expressions for the special case of SIMO/MISO and SISO systems respectively.

*Corollary 3:* The variance of the mutual information of SIMO/MISO-OFDM systems at high SNR is given by

$$\text{Var}^\infty(\mathcal{I}_{\text{ofdm}}) = (\log_2(e))^2 \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) \tilde{\varphi}(\rho_d) + \frac{\psi'(n)}{N} \right) \quad (31)$$

where

$$\tilde{\varphi}(\rho_d) = \begin{cases} 0 & \text{for } |\rho_d| = 0 \\ L_{i_2}(1 - |\rho_d|^2) - H^2(n-1) + 2 \sum_{b=1}^{n-1} \frac{H(b-1)}{b} \\ + \sum_{b=1}^{n-1} \frac{1}{b} \left( \left( \frac{|\rho_d|^{2b}-1}{|\rho_d|^2} \right)^b \ln(1 - |\rho_d|^2) - \sum_{t=0}^{b-2} \frac{\left( \frac{|\rho_d|^{2b}-1}{|\rho_d|^2} \right)^{t+1}}{b-t-1} \right) & \text{for } 0 < |\rho_d| < 1 \\ \psi'(n) & \text{for } |\rho_d| = 1 \end{cases} \quad (32)$$

where  $L_{i_2}(\cdot)$  is the *dilogarithm* function [36, eq. (27.7.1)] .  $\square$

*Corollary 4:* The variance of the mutual information of SISO-OFDM systems at high SNR is given by

$$\text{Var}^\infty(\mathcal{I}_{\text{ofdm}}) = (\log_2(e))^2 \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) L_{i_2}(1 - |\rho_d|^2) + \frac{\pi^2}{6N} \right) \quad (33)$$

$\square$

It is important to note that the results in Theorem 2 and Corollaries 3 and 4 do not depend on the SNR. Therefore, a main insight which we can draw from these expressions is that the variance of the MIMO-OFDM mutual information converges to a deterministic limit as the SNR increases, which we have now quantified precisely. This phenomenon is illustrated in Fig. 3, where we plot the variance of the MIMO-OFDM mutual information for different  $N_t \times N_r$  antenna configurations, and for different SNRs. The ‘‘Analytic Variance (High SNR)’’ dashed lines are based on (24) for the  $2 \times 3$  case, (31) for the  $1 \times 2$  case, and (33) for the  $1 \times 1$  case. The ‘‘Analytic Variance (Exact)’’ curves are based on (9) for the  $2 \times 3$  case, (20) for the  $1 \times 2$  case, and (22) for the  $1 \times 1$  case. Monte-Carlo simulated variance curves are also presented for further verification. We see that the results converge quickly in all cases.

### C. Analysis at Low SNR

The following theorem presents a very simple closed-form expression for the variance of the mutual information of MIMO-OFDM in the low SNR regime.

*Theorem 3:* In the low SNR regime, the variance of the mutual information of MIMO-OFDM systems is given by

$$\text{Var}^0(\mathcal{I}_{\text{ofdm}}) = (\log_2(e))^2 \frac{\gamma^2 N_r}{N N_t} \left( 1 + 2 \sum_{d=1}^{N-1} \frac{N-d}{N} |\rho_d|^2 \right). \quad (34)$$

*Proof:* See Appendix IV.  $\square$

The following corollary gives upper and lower bounds (as a function of the frequency correlation coefficients) for the variance of the MIMO-OFDM mutual information in the low SNR regime.

*Corollary 5:* In the low SNR regime, the variance of the mutual information of MIMO-OFDM systems satisfies

$$\frac{1}{N} \leq \frac{\text{Var}^0(\mathcal{I}_{\text{ofdm}})}{\text{Var}^0(\mathcal{I}_{\text{flat}})} = \frac{1}{N} \left( 1 + 2 \sum_{d=1}^{N-1} \frac{N-d}{N} |\rho_d|^2 \right) \leq 1 \quad (35)$$

where  $\text{Var}^0(\mathcal{I}_{\text{flat}})$  denotes the mutual information variance for an i.i.d. flat-fading Rayleigh MIMO channel. The left-hand side is an equality for  $|\rho_d| = 0$  (independent fading across all frequency subcarriers), and the right-hand side is an equality for  $|\rho_d| = 1$  (identical fading across all subcarriers, i.e. flat-fading).

*Proof:* The proof follows by using

$$0 \leq |\rho_d| \leq 1 \quad (36)$$

in (34), and noting that

$$\text{Var}^0(\mathcal{I}_{\text{flat}}) = (\log_2(e))^2 \frac{\gamma^2 N_r}{N_t}, \quad (37)$$

which is found by directly setting  $N = 1$  in (34).  $\square$

It is interesting to note from (35) that in the low SNR regime, the scaling of the MIMO-OFDM variance with respect to the flat-fading variance depends only on the channel delay profile, and is independent of the number of transmit and receive antennas.

For the particular case of a uniform power delay profile (i.e. with  $\sigma_p^2 = 1/L$  for all  $p = 0, \dots, L-1$ ), we can obtain a simple insightful expression for the variance ratio in (35), as

given below.

*Corollary 6:* For a uniform power delay profile, (35) becomes

$$\frac{1}{N} \leq \frac{\text{Var}^0(\mathcal{I}_{\text{ofdm}})}{\text{Var}^0(\mathcal{I}_{\text{flat}})} = \frac{1}{N} \left( 1 + 2 \sum_{d=1}^{N-1} \frac{N-d}{N} \left( \frac{\sin\left(\frac{\pi d L}{N}\right)}{L \sin\left(\frac{\pi d}{N}\right)} \right)^2 \right) \leq 1 \quad (38)$$

where the left-hand side is an equality for  $L = N$ , and the right-hand side is an equality for the case  $L = 1$ .

*Proof:* The proof follows trivially from (35) after noting that the frequency correlation-coefficients (5) in this case can be expressed as [37]

$$\rho_d = \frac{\sin\left(\frac{\pi d L}{N}\right)}{L \sin\left(\frac{\pi d}{N}\right)} e^{j \frac{\pi d}{N}}. \quad (39)$$

□

The summation in (38) is of a similar type to that in [33, eq. (60)], which gave an asymptotic expression for the variance for large antenna numbers, and involved the same squared-ratio terms. As mentioned in [33], as  $L$  increases, the ratio becomes more peaked as a function of  $d$ , thereby decreasing the overall sum. Thus, from (38) we see that the variance of the mutual information varies inversely with the channel delay spread in the low SNR regime. This agrees with previous observations seen via simulation studies in [19], and for the regime of large antenna numbers in [33]. These results are further corroborated in Fig. 4, where (38) is plotted as a function of the channel length  $L$ .

#### IV. OUTAGE APPROXIMATION OF MIMO-OFDM BASED SPATIAL-MULTIPLEXING

We now use the analytic expressions from the previous section to present and investigate approximations for the distribution of mutual information. We then use the approximations to estimate outage capacity.

Unless otherwise stated, for all results in this section we model the channel according to the

exponential power delay profile [38]

$$\sigma_p^2 = \begin{cases} \frac{1-e^{-1/K_{\text{exp}}}}{1-e^{-L/K_{\text{exp}}}} e^{-p/K_{\text{exp}}} & \text{for } 0 \leq p < L \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

where  $K_{\text{exp}}$  is a parameter which characterizes the rate of decay of the power delay profile as a function of  $p$ , and is loosely related to the rms delay spread [38].

#### A. Gaussian and Gamma Approximations

We first investigate the accuracy of a Gaussian approximation for various system configurations and channel scenarios.

Fig. 5 presents the analytical Gaussian approximation for the MIMO-OFDM mutual information p.d.f. based on the exact mean and variance expressions in (48) and (1) respectively, as well as empirically generated p.d.f.s (Monte-Carlo histogram), for different antenna configurations. A 64-subcarrier system is considered with SNR of 20 dB. We see that the analytic curves match the true distribution almost perfectly for both antenna configurations. We also present curves for a simulation based Gaussian approximation (based on the mean and variance of the Monte-Carlo generated histograms) for further verification. Note that these curves are indistinguishable from our new analytical Gaussian approximation curves.

Fig. 6 compares the analytical Gaussian approximation with empirically-generated p.d.f. curves, for different channel rms delay spreads. Again we see that the analytic Gaussian approximation is accurate in all cases. Moreover, we see a significant reduction in the variance of the mutual information as the rms delay spread increases (i.e. as  $K_{\text{exp}}$  increases). Again note that the Monte-Carlo Gaussian approximation is indistinguishable from our new analytical Gaussian approximation curves.

Fig. 7 compares our new analytic Gaussian approximation with the asymptotic Gaussian approximation previously derived in [33]; formally derived under the assumption of asymptotically large antenna numbers. To our knowledge, this is the only other comparable analytical result

in the literature which applies for arbitrary-length frequency-selective MIMO channels. In the figure, we consider a  $2 \times 2$  system at 20 dB SNR. The channel has a uniform power-delay profile; for which simple approximations to the mean and variance of the mutual information were explicitly presented in [33, eqs. (59) and (60)]. Clearly, although the approximation in [33] was shown to be quite accurate for some practical scenarios, Fig. 7 shows that our analytic Gaussian approximation is more accurate (although it is only shown for 32 subcarriers, the same observation has been made for all systems investigated).

Fig. 8 presents the distribution of the mutual information at high SNRs, comparing MIMO, SIMO, and SISO systems. The analytic Gaussian approximation curves are based on a high SNR mean formula from [3, Theorem 2], and the high SNR variance formula (24) for the MIMO case, (31) for the SIMO case, and (33) for the SISO case. We see that the analytic Gaussian approximation is accurate in all cases. Again note that the Monte-Carlo Gaussian approximation is indistinguishable from our new analytical Gaussian approximation curves.

Fig. 9 presents the distribution of the mutual information at low SNRs. The analytic Gaussian approximation curve is generated based on the low SNR mean formula obtained by combining (131) and (133), and the low SNR variance formula (34). In this case we see that a Gaussian distribution no longer accurately predicts the mutual information p.d.f. This can be explained by examining (131), where we see that at low SNRs the mutual information for each subcarrier is a function of  $\text{tr}(\mathbf{H}_k \mathbf{H}_k^\dagger)$ , which for i.i.d. Rayleigh fading is  $\sim \chi_{2N_r N_t}^2$ . Hence, the overall mutual information (8) is distributed as the sum of  $N$  correlated  $\chi_{2N_r N_t}^2$  random variables which (for small  $N$ ), is clearly quite different to Gaussian.

Motivated by this observation, we propose to approximate the mutual information p.d.f. at low SNR with a Gamma distribution. Note that a Gamma approximation was previously considered in the context of flat-fading channels in [39]. The Gamma p.d.f. is given by

$$f(x) = \frac{\theta^r x^{r-1} e^{-\theta x}}{\Gamma(r)}, \quad x \geq 0 \quad (41)$$

where  $r$  is the shape parameters and  $\theta$  is the scale parameter. By matching the first two moments, a Gamma approximation for the mutual information p.d.f. of MIMO-OFDM is obtained by evaluating

$$r = \frac{E[\mathcal{I}_{\text{ofdm}}]}{\text{Var}(\mathcal{I}_{\text{ofdm}})} \quad (42)$$

and

$$\theta = \frac{E^2[\mathcal{I}_{\text{ofdm}}]}{\text{Var}(\mathcal{I}_{\text{ofdm}})}. \quad (43)$$

This analytic Gamma approximation is plotted in Fig. 9, based on the same low SNR analytic mean and variance formulas as used for the low SNR Gaussian approximation above. We clearly see that the Gamma approximation is much more accurate than the Gaussian approximation in this low SNR regime, and follows the simulated p.d.f. very closely.

### B. Outage Capacity

The outage capacity  $\mathcal{I}_{\text{out},q}$  is defined as the maximum information rate guaranteed to be supported for  $100(1 - q)\%$  of the channel realizations<sup>3</sup>, ie.

$$P(\mathcal{I}_{\text{ofdm}} \leq \mathcal{I}_{\text{out},q}) = q \quad (44)$$

where  $q$  denotes the outage probability, and is thus directly obtained by inverting the c.d.f. of  $\mathcal{I}_{\text{ofdm}}$ . If the distribution of the mutual information is Gaussian, then the outage capacity can be computed from the derived mean and variance as [32, eq. (26)]

$$\mathcal{I}_{\text{out},q} = E[\mathcal{I}_{\text{ofdm}}] - \sqrt{\text{Var}(\mathcal{I}_{\text{ofdm}})}Q^{-1}(q) \quad (45)$$

where  $Q(\cdot)$  is the Gaussian  $Q$ -function.

Fig. 10 plots the outage probability for channels with different rms delay spreads. The ‘‘Analytic Gaus Approx’’ curves are generated by approximating the c.d.f. in (44) as a Gaussian distribution,

<sup>3</sup>Strictly speaking, computing the outage capacity would require performing a numerical optimization over all possible input distributions, as discussed in [1]. Here however, we adopt a common slight abuse of terminology, and use the term outage capacity to denote the outage rate for the case of OFDM-based spatial multiplexing systems with equal power Gaussian inputs.

and using the exact mean and variance formulas in (48) and (1) respectively. Clearly this analytic Gaussian approximation matches closely with the empirically generated c.d.f. (Monte-Carlo histogram) in all cases. Moreover, we see that for outage probabilities of practical interest (e.g.  $q = 1\%$ ), increasing the rms delay spread can yield a significant improvement in outage capacity.

## V. CONCLUSIONS

This paper has considered the mutual information distribution of frequency-selective MIMO channels, in the context of OFDM-based spatial multiplexing systems. Exact closed-form expressions were presented for the mutual information variance, applying for arbitrary finite system and channel parameters. These results were used to provide accurate analytical approximations for the distribution of mutual information, and the outage capacity. We observed that for most scenarios a Gaussian approximation is accurate, while also noting that for low SNR a Gamma approximation yielded even higher accuracy.

## APPENDIX I

### PROOF OF THEOREM 1

*Proof:* By definition, the variance of the mutual information is given by

$$\text{Var}(\mathcal{I}_{\text{ofdm}}) = E[\mathcal{I}_{\text{ofdm}}^2] - E^2[\mathcal{I}_{\text{ofdm}}] . \quad (46)$$

Noting that  $E[\mathcal{I}_{\text{ofdm}}] = E[\mathcal{I}_{\text{flat}}]$ , and using (6), we have

$$\begin{aligned} \text{Var}(\mathcal{I}_{\text{ofdm}}) &= E\left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \mathcal{I}_k \mathcal{I}_\ell\right] - E^2[\mathcal{I}_{\text{flat}}] \\ &= \frac{1}{N^2} \left( \sum_{k=0}^{N-1} \sum_{\ell=0, \ell \neq k}^{N-1} E[\mathcal{I}_k \mathcal{I}_\ell] + \sum_{k=0}^{N-1} E[\mathcal{I}_k^2] \right) - E^2[\mathcal{I}_{\text{flat}}] \\ &= \frac{1}{N^2} \left( \sum_{k=0}^{N-1} \sum_{\ell=0, \ell \neq k}^{N-1} E[\mathcal{I}_k \mathcal{I}_\ell] \right) + \frac{1}{N} E[\mathcal{I}_{\text{flat}}^2] - E^2[\mathcal{I}_{\text{flat}}] \end{aligned} \quad (47)$$

where  $\mathcal{I}_{\text{flat}}$  denotes the mutual information of a flat-fading channel. Note that the last line followed by noting that, under the assumptions in Section II-A, the channel statistics for each subcarrier (and therefore, the mutual information statistics) are identical [19], and moreover, these statistics are equal to that of a flat-fading i.i.d. Rayleigh channel. The first and second moments of the mutual information for flat-fading channels has been previously derived in terms of incomplete gamma functions in [15, eqs. (29) and (31)]. Using [36, eq. (6.5.9)], we perform some basic manipulations to express these results in alternative simplified forms as follows

$$E[\mathcal{I}_{\text{flat}}] = \frac{\log_2(e)}{\Gamma_m(n)\Gamma_m(m)} \sum_{r=1}^m \det(\mathbf{A}_r) \quad (48)$$

$$E[\mathcal{I}_{\text{flat}}^2] = \frac{(\log_2(e))^2}{\Gamma_m(n)\Gamma_m(m)} \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{B}_{r,s}) \quad (49)$$

where  $\mathbf{A}_r$  and  $\mathbf{B}_{r,s}$  are defined in (12) and (13) respectively.

The challenge is to evaluate the cross-correlation of the mutual information across frequency subcarriers  $E[\mathcal{I}_k \mathcal{I}_\ell]$  which, using (7), is given by

$$\begin{aligned} E[\mathcal{I}_k \mathcal{I}_\ell] &= E \left[ \log_2 \det \left( \mathbf{I}_{N_r} + \frac{\gamma}{N_t} \mathbf{H}_k \mathbf{H}_k^\dagger \right) \log_2 \det \left( \mathbf{I}_{N_r} + \frac{\gamma}{N_t} \mathbf{H}_\ell \mathbf{H}_\ell^\dagger \right) \right] \\ &= E \left[ \sum_{i=1}^m \log_2 \left( 1 + \frac{\gamma}{N_t} \lambda_i \right) \sum_{j=1}^m \log_2 \left( 1 + \frac{\gamma}{N_t} \omega_j \right) \right] \end{aligned} \quad (50)$$

where  $\underline{\lambda} = \{\lambda_i\}_{i=1}^m$  and  $\underline{\omega} = \{\omega_i\}_{i=1}^m$  are the non-zero eigenvalues of  $\mathbf{H}_k \mathbf{H}_k^\dagger$  and  $\mathbf{H}_\ell \mathbf{H}_\ell^\dagger$  respectively. Defining

$$\alpha(x) = \log_2 \left( 1 + \frac{\gamma}{N_t} x \right) \quad (51)$$

we have

$$E[\mathcal{I}_k \mathcal{I}_\ell] = E \left[ \sum_{i=1}^m \sum_{j=1}^m \alpha(\lambda_i) \alpha(\omega_j) \right] = \sum_{i=1}^m \sum_{j=1}^m E[\alpha(\lambda_i) \alpha(\omega_j)]. \quad (52)$$

Now, to evaluate the expectations in (52), we first simplify the problem by exploiting the symmetry with respect to the  $\lambda_i$ s and  $\omega_j$ s. To this end, let  $\lambda$  and  $\omega$  be randomly (uniformly)

chosen eigenvalues from  $\underline{\lambda}$  and  $\underline{\omega}$  respectively. Then clearly

$$\Pr(\lambda = \lambda_i, \omega = \omega_j) = \frac{1}{m^2}, \quad (53)$$

for any given  $i \in \{1, \dots, m\}, j \in \{1, \dots, m\}$ . Hence, we can also write

$$\begin{aligned} E[\alpha(\lambda)\alpha(\omega)] &= \sum_{i=1}^m \sum_{j=1}^m \Pr(\lambda = \lambda_i, \omega = \omega_j) E[\alpha(\lambda)\alpha(\omega)|\lambda = \lambda_i, \omega = \omega_j] \\ &= \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m E[\alpha(\lambda_i)\alpha(\omega_j)]. \end{aligned} \quad (54)$$

where the second line follows from (53). Therefore by directly comparing (54) with (52) it follows that

$$E[\mathcal{I}_k \mathcal{I}_\ell] = m^2 E[\alpha(\lambda)\alpha(\omega)]. \quad (55)$$

We point out that the simplification from (52) to (55) is particularly important, since in order to evaluate the expectation in (55), clearly we only require the distribution of a pair of arbitrarily-selected eigenvalues,  $\lambda$  and  $\omega$ . This turns out to be much more convenient than dealing with the distributions of the individual pairs of ordered eigenvalues, i.e.  $\lambda_i$  and  $\omega_j$ , required to directly evaluate (52).

The joint p.d.f. of  $\lambda$  and  $\omega$  is presented in Lemma 1 in Appendix II. From this lemma we see that  $f(\lambda, \omega)$ , and correspondingly  $E[\mathcal{I}_k \mathcal{I}_\ell]$  in (55), only depends on  $k$  and  $\ell$  through their absolute difference, i.e. since  $f(\lambda, \omega)$  only depends on  $k$  and  $\ell$  via  $|\rho_{k-\ell}|$ , and from (5)

$$|\rho_{k-\ell}| = |\rho_{\ell-k}^*| = |\rho_{\ell-k}|. \quad (56)$$

Therefore the left-hand summation in (47) can be written as

$$\sum_{k=0}^{N-1} \sum_{\ell=0, \ell \neq k}^{N-1} E[\mathcal{I}_k \mathcal{I}_\ell] = 2 \sum_{d=1}^{N-1} (N-d) E[\mathcal{I}_0 \mathcal{I}_d]. \quad (57)$$

Note that for subcarrier spacings  $d$  for which the frequency matrices are independent (i.e.  $\rho_d = 0$ ) or completely correlated (i.e.  $\rho_d = 1$ ), the expectations in (57) are evaluated trivially as

$$E[\mathcal{I}_0 \mathcal{I}_d] = E^2[\mathcal{I}_{\text{flat}}], \quad \rho_d = 0$$

$$E[\mathcal{I}_0 \mathcal{I}_d] = E[\mathcal{I}_{\text{flat}}^2], \quad \rho_d = 1. \quad (58)$$

For the case  $0 < |\rho_d| < 1$  such a direct evaluation is not possible, and we use (55) in Lemma 1 and (67) to evaluate the expectations in (57) as follows

$$\begin{aligned} E[\mathcal{I}_0 \mathcal{I}_d] &= m^2 \int_0^\infty \int_0^\infty \alpha(\lambda) \alpha(\omega) \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n) \Gamma_m(m) m^2 (1 - |\rho_d|^2)^m} \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{D}_{r,s}(\lambda, \omega)) d\lambda d\omega \\ &= \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n) \Gamma_m(m) (1 - |\rho_d|^2)^m} \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{D}_{r,s}), \quad 0 < |\rho_d| < 1 \end{aligned} \quad (59)$$

where  $\mathbf{D}_{r,s}$  is an  $m \times m$  matrix with  $(i, j)^{\text{th}}$  element

$$\left( \mathbf{D}_{r,s} \right)_{i,j} = \begin{cases} a(i, j) \triangleq \int_0^\infty \int_0^\infty \lambda^{\frac{\tau}{2}+i-1} \omega^{\frac{\tau}{2}+j-1} e^{-\frac{\lambda}{1-|\rho_d|^2}} e^{-\frac{\omega}{1-|\rho_d|^2}} \\ \quad \times I_\tau \left( \frac{2|\rho_d|}{1-|\rho_d|^2} \sqrt{\lambda\omega} \right) \alpha(\lambda) \alpha(\omega) d\lambda d\omega & \text{for } i = r, j = s \\ b(i, j) \triangleq \int_0^\infty \frac{\Gamma(\tau+j) |\rho_d|^\tau}{(1-|\rho_d|^2)^{-j}} e^{-\lambda} \lambda^{\tau+i-1} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2 \lambda}{1-|\rho_d|^2} \right)^t \frac{\alpha(\lambda)}{(\tau+t)!} d\lambda & \text{for } i = r, j \neq s \\ c(i, j) \triangleq \int_0^\infty \frac{\Gamma(\tau+i) |\rho_d|^\tau}{(1-|\rho_d|^2)^{-i}} e^{-\omega} \omega^{\tau+j-1} \sum_{t=0}^{i-1} \binom{i-1}{t} \left( \frac{|\rho_d|^2 \omega}{1-|\rho_d|^2} \right)^t \frac{\alpha(\omega)}{(\tau+t)!} d\omega & \text{for } i \neq r, j = s \\ \frac{\Gamma(\tau+j) |\rho_d|^\tau}{(1-|\rho_d|^2)^{-j}} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2}{1-|\rho_d|^2} \right)^t \frac{(\tau+i+t-1)!}{(\tau+t)!} & \text{for } i \neq r, j \neq s \end{cases} \quad (60)$$

Using the identity [11]

$$\int_0^\infty \ln(1 + \alpha\lambda) \lambda^{q-1} e^{-b\lambda} d\lambda = \Gamma(q) e^{b/\alpha} b^{-q} \sum_{h=1}^q E_h \left( \frac{b}{\alpha} \right) \quad (61)$$

we can evaluate  $b(i, j)$  and  $c(i, j)$  in closed-form as

$$b(i, j) = \frac{\log_2(e) e^{N_t/\gamma} \Gamma(\tau+j) |\rho_d|^\tau}{(1-|\rho_d|^2)^{-j}} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2}{1-|\rho_d|^2} \right)^t \frac{\Gamma(u)}{(\tau+t)!} g_1(u) \quad (62)$$

and

$$c(i, j) = \frac{\log_2(e) e^{N_t/\gamma} \Gamma(\tau+i) |\rho_d|^\tau}{(1-|\rho_d|^2)^{-i}} \sum_{t=0}^{i-1} \binom{i-1}{t} \left( \frac{|\rho_d|^2}{1-|\rho_d|^2} \right)^t \frac{\Gamma(v)}{(\tau+t)!} g_1(v) \quad (63)$$

respectively. We evaluate the remaining integral,  $a(i, j)$ , by using the power series expansion

$$I_\tau(x) = \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^{\tau+2k} \frac{1}{k! (\tau+k)!} \quad (64)$$

and integrating term by term using (61), to obtain

$$a(i, j) = (\log_2(e))^2 e^{\frac{2N_t}{\gamma(1-|\rho_d|^2)}} |\rho_d|^\tau (1 - |\rho_d|^2)^{\tau+i+j} \sum_{t=0}^{\infty} \frac{|\rho_d|^{2t} \Gamma(u)\Gamma(v)}{t! (\tau+t)!} g_2(u)g_2(v). \quad (65)$$

Substituting (62), (63), and (65) into (60), we perform some basic algebraic manipulations to write (59) as follows

$$E[\mathcal{I}_0 \mathcal{I}_d] = \frac{(\log_2(e))^2 e^{2N_t/\gamma}}{\Gamma_m(n)\Gamma_m(m)} \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{C}_{r,s}(\rho_d)) \quad . \quad 0 < |\rho_d| < 1. \quad (66)$$

The proof is completed by substituting (66) and (58) into (57), and then substituting (57), (49) and (48) into (47) and simplifying.

□

## APPENDIX II

### JOINT P.D.F. OF ARBITRARILY SELECTED EIGENVALUES OF SUBCARRIER MATRICES

*Lemma 1:* Let  $\lambda$  and  $\omega$  be arbitrarily selected non-zero eigenvalues of the subcarrier channel matrices  $\mathbf{H}_k \mathbf{H}_k^\dagger$  and  $\mathbf{H}_\ell \mathbf{H}_\ell^\dagger$  respectively. Then the joint p.d.f. of  $\lambda$  and  $\omega$  is given by

$$f(\lambda, \omega) = \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n)\Gamma_m(m)m^2(1-|\rho_d|^2)^m} \sum_{r=1}^m \sum_{s=1}^m \det(\mathbf{D}_{r,s}(\lambda, \omega)) \quad (67)$$

where  $d = k - \ell$ ,  $\tau = n - m$ , and  $\mathbf{D}_{r,s}(\lambda, \omega)$  is an  $m \times m$  matrix with  $(i, j)$ <sup>th</sup> element

$$(\mathbf{D}_{r,s}(\lambda, \omega))_{i,j} = \begin{cases} \lambda^{\frac{\tau}{2}+i-1} \omega^{\frac{\tau}{2}+j-1} e^{-\frac{\lambda}{1-|\rho_d|^2}} e^{-\frac{\omega}{1-|\rho_d|^2}} I_\tau \left( \frac{2|\rho_d|}{1-|\rho_d|^2} \sqrt{\lambda\omega} \right) & \text{for } i = r, j = s \\ \frac{\Gamma(\tau+j)|\rho_d|^\tau}{(1-|\rho_d|^2)^{-j}} e^{-\lambda} \lambda^{\tau+i-1} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2 \lambda}{1-|\rho_d|^2} \right)^t \frac{1}{(\tau+t)!} & \text{for } i = r, j \neq s \\ \frac{\Gamma(\tau+i)|\rho_d|^\tau}{(1-|\rho_d|^2)^{-i}} e^{-\omega} \omega^{\tau+j-1} \sum_{t=0}^{i-1} \binom{i-1}{t} \left( \frac{|\rho_d|^2 \omega}{1-|\rho_d|^2} \right)^t \frac{1}{(\tau+t)!} & \text{for } i \neq r, j = s \\ \frac{\Gamma(\tau+j)|\rho_d|^\tau}{(1-|\rho_d|^2)^{-j}} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2}{1-|\rho_d|^2} \right)^t \frac{(\tau+i+t-1)!}{(\tau+t)!} & \text{for } i \neq r, j \neq s \end{cases} \quad (68)$$

where  $I_\tau(\cdot)$  is the modified Bessel function of the first kind [36, eq. (9.6.10)].

*Proof:* From (5), we see that  $\mathbf{H}_k \mathbf{H}_k^\dagger$  and  $\mathbf{H}_\ell \mathbf{H}_\ell^\dagger$  are (frequency) *correlated Wishart matrices*. In [40], the joint ordered eigenvalue density for matrices of this general form was evaluated for cases where the correlation coefficient was real. Extending this result to complex correlation

coefficients, and to unordered eigenvalues, we obtain the joint eigenvalue density

$$f_u(\underline{\lambda}, \underline{\omega}) = \frac{1}{m!^2 \Gamma_m(n) \Gamma_m(m)} \frac{|\rho_d|^{-m(n-1)}}{(1 - |\rho_d|^2)^m} \exp\left(-\frac{\sum_{t=1}^m (\lambda_t + \omega_t)}{1 - |\rho_d|^2}\right) \Delta_m(\underline{\lambda}) \Delta_m(\underline{\omega}) \\ \times \det\left((\lambda_i \omega_j)^{\frac{\tau}{2}} I_\tau\left(\frac{2|\rho_d|}{1 - |\rho_d|^2} \sqrt{\lambda_i \omega_j}\right)\right) \quad (69)$$

where  $\Delta_m(\cdot)$  is a Vandermonde determinant, defined as

$$\Delta_m(\underline{\lambda}) = \prod_{i < j}^m (\lambda_j - \lambda_i) = \det(\lambda_i^{j-1}) . \quad (70)$$

Note that the extension from ordered to unordered eigenvalues simply involved the addition of the leading  $1/m!^2$  factor in (69), whereas the extension from real to complex correlation coefficients is trivial, and the proof is omitted.

To evaluate (67) we marginalize (69) as follows

$$f(\lambda, \omega) = \int_{\lambda_2} \cdots \int_{\lambda_m} \int_{\omega_2} \cdots \int_{\omega_m} f_u(\underline{\lambda}, \underline{\omega}) d\lambda_2 \cdots d\lambda_m d\omega_2 \cdots d\omega_m \quad (71)$$

where we have let  $\lambda_1 = \lambda$  and  $\omega_1 = \omega$ . We evaluate these integrals by first expanding the Vandermonde determinants in (69) according to

$$\Delta_m(\underline{\lambda}) \Delta_m(\underline{\omega}) = \sum_{\alpha} (-1)^{\text{per}(\alpha)} \prod_{i=1}^m \lambda_i^{\alpha_i - 1} \sum_{\beta} (-1)^{\text{per}(\beta)} \prod_{j=1}^m \omega_j^{\beta_j - 1} \quad (72)$$

where the sums are over all permutations  $\alpha = \{\alpha_1, \dots, \alpha_m\}$  and  $\beta = \{\beta_1, \dots, \beta_m\}$  of  $\{1, \dots, m\}$ , and  $(-1)^{\text{per}(\alpha)}$  and  $(-1)^{\text{per}(\beta)}$  denote the signs of the permutations. Substituting (72) and (69) into (71) yields

$$f(\lambda, \omega) = \int_{\lambda_2} \cdots \int_{\omega_m} \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n) \Gamma_m(m) (m!)^2 (1 - |\rho_d|^2)^m} \exp\left(-\frac{\sum_{t=1}^m (\lambda_t + \omega_t)}{1 - |\rho_d|^2}\right) \sum_{\alpha} (-1)^{\text{per}(\alpha)} \\ \times \prod_{i=1}^m \lambda_i^{\alpha_i - 1} \sum_{\beta} (-1)^{\text{per}(\beta)} \prod_{j=1}^m \omega_j^{\beta_j - 1} \det\left((\lambda_i \omega_j)^{\frac{\tau}{2}} I_\tau\left(\frac{2|\rho_d|}{1 - |\rho_d|^2} \sqrt{\lambda_i \omega_j}\right)\right) d\lambda_2 \cdots d\omega_m \\ = \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n) \Gamma_m(m) (m!)^2 (1 - |\rho_d|^2)^m} \sum_{\alpha} \sum_{\beta} (-1)^{\text{per}(\alpha) + \text{per}(\beta)} \\ \times \int_{\lambda_2} \cdots \int_{\omega_m} \det(a(\lambda_i, \omega_j, \alpha_i, \beta_j)) d\lambda_2 \cdots d\omega_m \quad (73)$$

where

$$a(\lambda_i, \omega_j, \alpha_i, \beta_j) = \lambda_i^{\frac{\tau}{2} + \alpha_i - 1} \omega_j^{\frac{\tau}{2} + \beta_j - 1} e^{-\frac{\lambda_i}{1 - |\rho_d|^2}} e^{-\frac{\omega_j}{1 - |\rho_d|^2}} I_\tau \left( \frac{2|\rho_d|}{1 - |\rho_d|^2} \sqrt{\lambda_i \omega_j} \right). \quad (74)$$

Expanding the determinants, integrating term by term, and re-forming determinants, we obtain

$$f(\lambda, \omega) = \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n)\Gamma_m(m)(m!)^2(1 - |\rho_d|^2)^m} \sum_{\alpha} \sum_{\beta} (-1)^{\text{per}(\alpha) + \text{per}(\beta)} \det \left( \tilde{\mathbf{D}}_{\alpha, \beta}(\lambda, \omega) \right) \quad (75)$$

where  $\tilde{\mathbf{D}}_{\alpha, \beta}(\cdot)$  is an  $m \times m$  matrix with  $(i, j)$ th element

$$\left( \tilde{\mathbf{D}}_{\alpha, \beta}(\lambda, \omega) \right)_{i, j} = \begin{cases} a(\lambda, \omega, \alpha_i, \beta_j) & \text{for } i = 1, j = 1 \\ b(\lambda, \alpha_i, \beta_j) \triangleq \int_0^\infty a(\lambda, \omega_j, \alpha_i, \beta_j) d\omega_j & \text{for } i = 1, j \neq 1 \\ c(\omega, \alpha_i, \beta_j) \triangleq \int_0^\infty a(\lambda_i, \omega, \alpha_i, \beta_j) d\lambda_i & \text{for } i \neq 1, j = 1 \\ d(\alpha_i, \beta_j) \triangleq \int_0^\infty \int_0^\infty a(\lambda_i, \omega_j, \alpha_i, \beta_j) d\lambda_i d\omega_j & \text{for } i \neq 1, j \neq 1 \end{cases} \quad (76)$$

Reordering rows and columns yields

$$\det \left( \tilde{\mathbf{D}}_{\alpha, \beta}(\lambda, \omega) \right) = (-1)^{\text{per}(\alpha) + \text{per}(\beta)} \det \left( \mathbf{D}_{\alpha_1, \beta_1}(\lambda, \omega) \right) \quad (77)$$

where

$$\left( \mathbf{D}_{\alpha_1, \beta_1}(\lambda, \omega) \right)_{i, j} = \begin{cases} a(\lambda, \omega, i, j) & \text{for } i = \alpha_1, j = \beta_1 \\ b(\lambda, i, j) & \text{for } i = \alpha_1, j \neq \beta_1 \\ c(\omega, i, j) & \text{for } i \neq \alpha_1, j = \beta_1 \\ d(i, j) & \text{for } i \neq \alpha_1, j \neq \beta_1 \end{cases}. \quad (78)$$

Applying (77) in (75) we can further simplify as follows

$$\begin{aligned} f(\lambda, \omega) &= \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n)\Gamma_m(m)(m!)^2(1 - |\rho_d|^2)^m} \sum_{\alpha} \sum_{\beta} \det \left( \mathbf{D}_{\alpha_1, \beta_1}(\lambda, \omega) \right) \\ &= \frac{|\rho_d|^{-m(n-1)} ((m-1)!)^2}{\Gamma_m(n)\Gamma_m(m)(m!)^2(1 - |\rho_d|^2)^m} \sum_{\alpha_1=1}^m \sum_{\beta_1=1}^m \det \left( \mathbf{D}_{\alpha_1, \beta_1}(\lambda, \omega) \right) \\ &= \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n)\Gamma_m(m)m^2(1 - |\rho_d|^2)^m} \sum_{r=1}^m \sum_{s=1}^m \det \left( \mathbf{D}_{r, s}(\lambda, \omega) \right). \end{aligned} \quad (79)$$

The result now follows by combining (74), (78) and (79), and by evaluating the integrals  $b(\cdot)$ ,

$c(\cdot)$  and  $d(\cdot)$  inside the remaining determinant, using the identities [40]

$$\int_0^\infty x^{a+\frac{t}{2}-1} e^{-cx} I_t(2\sqrt{fx}) dx = \frac{(t+a-1)!}{c^{\frac{t}{2}+a}} \left(\frac{f}{c}\right)^{\frac{t}{2}} e^{\frac{f}{c}} \sum_{r=0}^{a-1} \binom{a-1}{r} \frac{\left(\frac{f}{c}\right)^r}{(t+r)!} \quad (80)$$

for integers  $a$  and  $t$ , and [35]

$$\int_0^\infty x^t e^{-ax} dx = \Gamma(t+1) a^{-(t+1)} \quad (81)$$

for integer  $t \geq 0$ .

□

### APPENDIX III

#### PROOF OF THEOREM 2

*Proof:* We start by noting that at high SNR, (7) can be written as

$$\mathcal{I}_k = \log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_k \right) \quad (82)$$

where is an  $m \times m$  complex Wishart matrix given by

$$\mathbf{W}_k = \begin{cases} \mathbf{H}_k \mathbf{H}_k^\dagger & \text{for } N_r \leq N_t \\ \mathbf{H}_k^\dagger \mathbf{H}_k & \text{for } N_r > N_t \end{cases}. \quad (83)$$

Substituting (82) into (47) and using (57), we write the variance of the MIMO-OFDM mutual information at high SNR as follows

$$\begin{aligned} \text{Var}^\infty(\mathcal{I}_{\text{ofdm}}) &= \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) E \left[ \log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_0 \right) \log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_d \right) \right] \right. \\ &\quad \left. + \frac{1}{N} E \left[ \left( \log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_0 \right) \right)^2 \right] - E^2 \left[ \log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_0 \right) \right] \right). \quad (84) \end{aligned}$$

Noting that

$$\log_2 \det \left( \frac{\gamma}{N_t} \mathbf{W}_0 \right) = m \log_2 \left( \frac{\gamma}{N_t} \right) + \log_2 \det (\mathbf{W}_0) \quad (85)$$

we apply some simple algebra to (84) and find that the terms involving  $\gamma$  cancel perfectly, leaving

$$\text{Var}^\infty(\mathcal{I}_{\text{ofdm}}) = \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) E [\log_2 \det (\mathbf{W}_0) \log_2 \det (\mathbf{W}_d)] \right)$$

$$+ \frac{1}{N} E [(\log_2 \det (\mathbf{W}_0))^2] - E^2 [\log_2 \det (\mathbf{W}_0)] \Big) \quad (86)$$

Since  $\mathbf{W}_0$  is a complex Wishart matrix, we invoke results from [3] to give

$$E [\log_2 \det (\mathbf{W}_0)] = \log_2(e) \sum_{t=0}^{m-1} \psi(n-t) \quad (87)$$

$$E [(\log_2 \det (\mathbf{W}_0))^2] = (\log_2(e))^2 \left( \sum_{t=0}^{m-1} \psi'(n-t) + \sum_{t=0}^{m-1} \psi(n-t) \right). \quad (88)$$

We now consider the remaining expectation  $E [\log_2 \det (\mathbf{W}_0) \log_2 \det (\mathbf{W}_d)]$  in (86). For the extreme cases of  $\rho_d = 0$  and  $\rho_d = 1$ , this is directly obtained from (87) and (88) respectively.

The main challenge is to obtain a closed-form finite sum expression for  $0 < |\rho_d| < 1$ .

We start by following the same procedure as used in (50)-(60) in the proof of Theorem 1, which yields

$$\begin{aligned} E [\log_2 \det (\mathbf{W}_0) \log_2 \det (\mathbf{W}_d)] \\ = \frac{|\rho_d|^{-m(n-1)}}{\Gamma_m(n) \Gamma_m(m) (1 - |\rho_d|^2)^m} \sum_{r=1}^m \sum_{s=1}^m \det (\bar{\mathbf{D}}_{r,s}), \quad 0 < |\rho_d| < 1 \end{aligned} \quad (89)$$

where  $\bar{\mathbf{D}}_{r,s}$  is an  $m \times m$  matrix with entries corresponding to (60), but with the  $\alpha(\cdot)$  functions replaced with

$$\tilde{\alpha}(x) = \log_2(x). \quad (90)$$

We now evaluate the integrals for the elements of  $\bar{\mathbf{D}}_{r,s}$  corresponding to  $b(i, j)$  and  $c(i, j)$  in (60), using the identity [35, eq. (4.352.1)]

$$\int_0^\infty x^{q-1} e^{-bx} \ln(x) dx = \frac{\Gamma(q)}{b^q} (\psi(q) - \ln(b)). \quad q > 0, \quad b > 0 \quad (91)$$

This gives

$$b(i, j) = \frac{\log_2(e) \Gamma(\tau + j) |\rho_d|^\tau}{(1 - |\rho_d|^2)^{-j}} \sum_{t=0}^{j-1} \binom{j-1}{t} \left( \frac{|\rho_d|^2}{1 - |\rho_d|^2} \right)^t \frac{\Gamma(u) \psi(u)}{(\tau + t)!} \quad (92)$$

and

$$c(i, j) = \frac{\log_2(e) \Gamma(\tau + i) |\rho_d|^\tau}{(1 - |\rho_d|^2)^{-i}} \sum_{t=0}^{i-1} \binom{i-1}{t} \left( \frac{|\rho_d|^2}{1 - |\rho_d|^2} \right)^t \frac{\Gamma(v) \psi(v)}{(\tau + t)!}. \quad (93)$$

To evaluate the remaining integrals in  $\bar{\mathbf{D}}_{r,s}$ , i.e. for the elements  $a(i, j)$ , we use (64) and (91) to obtain

$$a(i, j) = (\log_2(e))^2 |\rho_d|^\tau (1 - |\rho_d|^2)^{\tau+i+j} \quad (94)$$

$$\times \sum_{t=0}^{\infty} \frac{|\rho_d|^{2t} \Gamma(u) \Gamma(v)}{t! (\tau+t)!} (H(u-1) + h(\rho_d)) (H(v-1) + h(\rho_d)) . \quad (95)$$

Next we use (92)-(94) in (89), and perform some basic simplifications to obtain

$$E [\log_2 \det (\mathbf{W}_0) \log_2 \det (\mathbf{W}_d)] = \frac{(\log_2(e))^2}{\Gamma_m(n) \Gamma_m(m)} \sum_{r=1}^m \sum_{s=1}^m \det (\tilde{\mathbf{C}}_{r,s}(\rho_d)) \quad (96)$$

where  $\tilde{\mathbf{C}}_{r,s}(\rho_d)$  is an  $m \times m$  matrix with  $(i, j)^{\text{th}}$  element

$$\left( \tilde{\mathbf{C}}_{r,s}(\rho_d) \right)_{i,j} = \begin{cases} \eta_{i,j}(1, \rho_d) & \text{for } i \neq r, j \neq s \\ \eta_{i,j}(\psi(z), \rho_d) & \text{for } i = r, j \neq s \\ |\rho_d|^{2(i-j)} \eta_{j,i}(\psi(z), \rho_d) & \text{for } i \neq r, j = s \\ \frac{(1-|\rho_d|^2)^z}{|\rho_d|^{2(j-1)}} \sum_{t=0}^{\infty} \frac{|\rho_d|^{2t} \Gamma(u) \Gamma(v) (h(\rho_d) + H(u-1)) (h(\rho_d) + H(v-1))}{t! (\tau+t)!} & \text{for } i = r, j = s \end{cases} \quad (97)$$

The expression (24) follows by using (96), (88), and (87) in (86).

To complete the proof we must express the infinite summation in (97) in the simplified finite-sum form of (27). This simplification requires significant algebraic manipulations, which we now detail. Start by recalling the definitions  $u = t + \tau + i$  and  $v = t + \tau + j$ , and writing the infinite sum in (97) as follows

$$\left( \tilde{\mathbf{C}}_{r,s}(\rho_d) \right)_{i,j} = \frac{(1 - |\rho_d|^2)^z}{|\rho_d|^{2(j-1)}} \mathcal{S}(|\rho_d|^2) \quad (98)$$

where

$$\mathcal{S}(x) \triangleq \sum_{t=0}^{\infty} \frac{x^t (\tau+t+i-1)! (\tau+t+i-1)!}{t! (\tau+t)!} \times (h(\sqrt{x}) + H(\tau+t+i-1)) (h(\sqrt{x}) + H(\tau+t+j-1)) . \quad (99)$$

Note that the series (99), and those that follow below, are convergent for  $|x| < 1$  (a condition which holds in (98)).

Now, (99) can be written as

$$\mathcal{S}(x) = h^2(\sqrt{x})\mathcal{S}_1(1, 1, x) + h(\sqrt{x}) (\mathcal{S}_1(H(i), 1, x) + \mathcal{S}_1(1, H(j), x)) + \mathcal{S}_1(H(i), H(j), x) \quad (100)$$

where

$$\mathcal{S}_1(f_1(i), f_2(j), x) \triangleq \sum_{t=0}^{\infty} \frac{x^t(\tau+t+i-1)!(\tau+t+j-1)!}{t!(\tau+t)!} f_1(\tau+t+i-1)f_2(\tau+t+j-1) \quad (101)$$

for arbitrary functions  $f_1$  and  $f_2$ . We now consider each of the infinite sums in (100) in turn.

First consider  $\mathcal{S}_1(1, 1, x)$ . Following a similar general approach to that used in [41], we perform the following sequence of operations<sup>4</sup>

$$\begin{aligned} \mathcal{S}_1(1, 1, x) &= \sum_{t=0}^{\infty} \frac{x^t(\tau+t+i-1)!(\tau+t+j-1)!}{t!(\tau+t)!} \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} \sum_{t=0}^{\infty} \frac{x^{t+\tau+i-1}(\tau+t+j-1)!}{(\tau+t)!} \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} \sum_{t=\tau}^{\infty} \frac{x^{t+i-1}(t+j-1)!}{t!} \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} \sum_{t=0}^{\infty} \frac{x^{t+i-1}(t+j-1)!}{t!} \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} x^{i-1} \sum_{t=0}^{\infty} \frac{x^t(t+j-1)!}{t!} \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} \left( x^{i-1} \frac{d^{j-1}}{dx^{j-1}} \sum_{t=0}^{\infty} x^{t+j-1} \right) \\ &= \frac{d^{\tau+i-1}}{dx^{\tau+i-1}} \left( x^{i-1} \frac{d^{j-1}}{dx^{j-1}} \sum_{t=0}^{\infty} x^t \right). \end{aligned} \quad (102)$$

Via application of the Leibnitz formula, it can be shown that

$$\mathcal{S}_1(1, 1, x) = \sum_{b=0}^{i-1} \binom{\tau+i-1}{\tau+b} \frac{(i-1)!}{b!} x^b \frac{d^{\tau+j+b-1}}{dx^{\tau+j+b-1}} \left( \sum_{t=0}^{\infty} x^t \right)$$

<sup>4</sup>Note that for this particular case, a finite expression could be also found by directly matching the infinite series to a hypergeometric function, and using associated identities; something which cannot be done in the other cases.

$$= \Gamma(\tau + i) \sum_{b=0}^{i-1} \binom{i-1}{b} \frac{x^b}{(\tau + b)!} \frac{d^{\tau+j+b-1}}{dx^{\tau+j+b-1}} \left( \sum_{t=0}^{\infty} x^t \right). \quad (103)$$

Now noting that

$$\sum_{t=0}^{\infty} x^t = \frac{1}{1-x}, \quad |x| < 1 \quad (104)$$

with derivatives

$$\frac{d^r}{dx^r} \left( \sum_{t=0}^{\infty} x^t \right) = \frac{r!}{(1-x)^{r+1}} \quad (105)$$

we can write (103) as follows

$$\begin{aligned} \mathcal{S}_1(1, 1, x) &= \frac{\Gamma(\tau + i)}{(1-x)^{\tau+j}} \sum_{b=0}^{i-1} \binom{i-1}{b} \left( \frac{x}{1-x} \right)^b \frac{(\tau + j + b - 1)!}{(\tau + b)!} \\ &= \frac{\Gamma(\tau + i) x^{i-1}}{(1-x)^z} \sum_{b=0}^{i-1} \binom{i-1}{b} \left( \frac{1-x}{x} \right)^b (\tau + i - b)_{j-1} \\ &= \frac{x^{i-1} \eta_{j,i}(1, \sqrt{x})}{(1-x)^z}. \end{aligned} \quad (106)$$

Now consider  $\mathcal{S}_1(H(i), 1, x)$ . Following the same sequence of operations as in (102) and (103),

we find that

$$\begin{aligned} \mathcal{S}_1(H(i), 1, x) &= \frac{d^{\tau+j-1}}{dx^{\tau+j-1}} \left( x^{j-1} \frac{d^{i-1}}{dx^{i-1}} \sum_{t=0}^{\infty} x^t H(t) \right) \\ &= \Gamma(\tau + j) \sum_{b=0}^{j-1} \binom{j-1}{b} \frac{x^b}{(\tau + b)!} \frac{d^{\tau+i+b-1}}{dx^{\tau+i+b-1}} \left( \sum_{t=0}^{\infty} x^t H(t) \right). \end{aligned} \quad (107)$$

Now we use [42]

$$\sum_{t=0}^{\infty} x^t H(t) = \frac{-\ln(1-x)}{1-x}, \quad |x| < 1 \quad (108)$$

and the corresponding derivatives

$$\frac{d^r}{dx^r} \left( \sum_{t=0}^{\infty} x^t H(t) \right) = \frac{r!}{(1-x)^{r+1}} (H(r) - \ln(1-x)), \quad (109)$$

to write (107) as follows

$$\mathcal{S}_1(H(i), 1, x) = \frac{\Gamma(\tau + j)}{(1-x)^{\tau+i}} \sum_{b=0}^{j-1} \binom{j-1}{b} \left( \frac{x}{1-x} \right)^b \frac{(\tau + i + b - 1)!}{(\tau + b)!} (H(\tau + i + b - 1) - \ln(1-x))$$

$$\begin{aligned}
&= \frac{x^{j-1}\Gamma(\tau+j)}{(1-x)^z} \sum_{b=0}^{j-1} \binom{j-1}{b} \left(\frac{1-x}{x}\right)^b (\tau+j-b)_{i-1} (H(z-1-b) - \ln(1-x)) \\
&= \frac{x^{j-1}}{(1-x)^z} (\eta_{i,j}(H(z-1), \sqrt{x}) - \ln(1-x)\eta_{i,j}(1, \sqrt{x})) . \tag{110}
\end{aligned}$$

Now consider  $\mathcal{S}_1(1, H(j), x)$ . Using exactly the same approach as for  $\mathcal{S}_1(H(i), 1, x)$ , we obtain

$$\mathcal{S}_1(1, H(j), x) = \frac{x^{i-1}}{(1-x)^z} (\eta_{j,i}(H(z-1), \sqrt{x}) - \ln(1-x)\eta_{j,i}(1, \sqrt{x})) . \tag{111}$$

Finally consider  $\mathcal{S}_1(H(i), H(j), x)$ . We follow the same sequence of operations as in (102) and (103). In this case it is convenient to take the successive derivatives based on the order of  $i$  and  $j$ . In particular, with  $i' = \max(i, j)$  and  $j' = \min(i, j)$ , we obtain

$$\begin{aligned}
\mathcal{S}_1(H(i), H(j), x) &= \frac{d^{\tau+i'-1}}{dx^{\tau+i'-1}} \left( x^{i'-1} \frac{d^{j'-1}}{dx^{j'-1}} \mathcal{S}_2(x) \right) \\
&= \Gamma(\tau+i') \sum_{b=0}^{i'-1} \binom{i'-1}{b} \frac{x^b}{(\tau+b)!} \frac{d^{\tau+j'+b-1}}{dx^{\tau+j'+b-1}} \mathcal{S}_2(x) \tag{112}
\end{aligned}$$

where

$$\mathcal{S}_2(x) \triangleq \sum_{t=0}^{\infty} x^t H(t) H(t+i'-j') . \tag{113}$$

In this case, in contrast to the previous summations in (104) and (108), the infinite summation in (113) cannot be directly expressed in a finite form. To evaluate this series in finite form, we start by using (29) to write

$$\begin{aligned}
\mathcal{S}_2(x) &= \sum_{t=1}^{\infty} x^t H(t) \left( H(t) + \sum_{q=1}^{i'-j'} \frac{1}{t+q} \right) \\
&= \sum_{t=1}^{\infty} x^t H(t)^2 + \mathcal{S}_3(x) \\
&= \frac{L_{i_2}(1-x) + \ln^2(1-x)}{1-x} + \mathcal{S}_3(x) \tag{114}
\end{aligned}$$

where  $L_{i_2}(\cdot)$  is the *dilogarithm* function [36, eq. (27.7.1)], and  $\mathcal{S}_3(\cdot)$  is given by

$$\mathcal{S}_3(x) \triangleq \sum_{q=1}^{i'-j'} \sum_{t=1}^{\infty} \frac{x^t H(t)}{t+q} . \tag{115}$$

Note that the last line in (114) followed by using an identity from [42]. We now manipulate

$\mathcal{S}_3(\cdot)$  as follows

$$\mathcal{S}_3(x) = \sum_{q=1}^{i'-j'} \frac{1}{x^q} \sum_{t=1}^{\infty} \frac{x^{t+q} H(t)}{t+q} \quad (116)$$

$$\begin{aligned} &= \sum_{q=1}^{i'-j'} \frac{1}{x^q} \sum_{t=1}^{\infty} \int x^{t+q-1} H(t) dx \\ &= \sum_{q=1}^{i'-j'} \frac{1}{x^q} \int x^{q-1} \left( \sum_{t=1}^{\infty} x^t H(t) \right) dx \\ &= \sum_{q=1}^{i'-j'} -\frac{1}{x^q} \int \frac{x^{q-1} \ln(1-x)}{1-x} dx. \end{aligned} \quad (117)$$

For  $q > 1$ , consider

$$\begin{aligned} \frac{x^{q-1}}{1-x} &= -x^{q-2} + \frac{x^{q-2}}{1-x} = \dots \\ &= \frac{1}{1-x} - \sum_{v=1}^{q-1} x^{v-1}, \quad q > 1, \end{aligned} \quad (118)$$

so therefore

$$\mathcal{S}_3(x) = \sum_{q=2}^{i'-j'} \frac{1}{x^q} \sum_{v=1}^{q-1} \int x^{v-1} \ln(1-x) dx - \sum_{q=1}^{i'-j'} \frac{1}{x^q} \int \frac{\ln(1-x)}{1-x} dx. \quad (119)$$

Using [35, Eq. 2.729]<sup>5</sup>

$$\int y^m \ln(1-y) dy = \frac{1}{m+1} \left( (y^{m+1} - 1) \ln(1-y) - \sum_{k=1}^{m+1} \frac{y^{m-k+2}}{m-k+2} \right) + \text{const} \quad (120)$$

and noting that

$$\begin{aligned} \int \frac{\ln(1-x)}{1-x} dx &= - \int \ln(1-x) \frac{d}{dx} \ln(1-x) dx \\ &= -\frac{\ln^2(1-x)}{2} + \text{const} \end{aligned} \quad (121)$$

we can now express  $\mathcal{S}_3(x)$  in finite form as follows

$$\mathcal{S}_3(x) = \sum_{q=2}^{i'-j'} \frac{1}{x^q} \sum_{v=1}^{q-1} \frac{1}{v} \left( (x^v - 1) \ln(1-x) - \sum_{t=1}^v \frac{x^t}{t} \right) + \frac{\ln^2(1-x)}{2} \sum_{q=1}^{i'-j'} \frac{1}{x^q}. \quad (122)$$

Note that it can be easily verified, using (116), that the integration constant generated in going from (119) to (122) is zero. After much algebraic manipulation, it can be shown that (122)

<sup>5</sup>There is a missing  $(-1)$  factor in this reference.

reduces to

$$\mathcal{S}_3(x) = \frac{\ln(1-x)}{2} \sum_{q=1}^{i'-j'} \frac{\ln(1-x)}{x^q} + \sum_{q=1}^{i'-j'-1} \left( \frac{\ln(1-x)H(i'-j'-q)}{x^q} - \frac{\ln(1-x)H(q)}{x^{q+1}} - \frac{1}{x^q} \sum_{r=1}^{i'-j'-q} \frac{H(r+q-1) - H(r-1)}{r} \right). \quad (123)$$

Now substituting (123) into (114) we can express  $\mathcal{S}_2(x)$  as the finite sum

$$\mathcal{S}_2(x) = \frac{L_{i_2}(1-x) + \ln^2(1-x)}{1-x} + \frac{\ln(1-x)}{2} \sum_{q=1}^{i'-j'} \frac{\ln(1-x)}{x^q} + \sum_{q=1}^{i'-j'-1} \left( \frac{\ln(1-x)H(i'-j'-q)}{x^q} - \frac{\ln(1-x)H(q)}{x^{q+1}} - \frac{1}{x^q} \sum_{r=1}^{i'-j'-q} \frac{H(r+q-1) - H(r-1)}{r} \right). \quad (124)$$

The corresponding derivatives can be obtained after tedious algebra as follows

$$\frac{d^r}{dx^r} \mathcal{S}_2(x) = \frac{r!}{(1-x)^{r+1}} \xi_x(r) \quad (125)$$

where

$$\begin{aligned} \xi_x(r) = & L_{i_2}(1-x) + \ln^2(1-x) - 2H(r) \ln(1-x) + \sum_{b=1}^r \left( \frac{2H(b-1) - f_{1,b-1}(x)}{b} \right) \\ & + \frac{1}{2} \sum_{q=1}^{\delta} \left( \ln(1-x) f_{q,r}(x) - \sum_{b=0}^{r-1} \frac{f_{q,b}(x)}{r-b} \right) \\ & + \sum_{q=1}^{\delta-1} \left( H(\delta-q) f_{q,r}(x) - H(q) f_{q+1,r}(x) + \mu_{q,r}(x) K(q) \right) \end{aligned} \quad (126)$$

where  $\delta = i' - j'$ , and recall that  $L_{i_2}(\cdot)$  is the *dilogarithm* function [36, eq. (27.7.1)]. Also,  $K(\cdot)$

is a constant given by

$$K(q) = \sum_{t=1}^{\delta-q} \frac{H(t+q-1) - H(t-1)}{t}, \quad (127)$$

and

$$f_{q,r}(x) = \sum_{t=0}^{r-1} \frac{\mu_{q,t}(x)}{r-t} - \mu_{q,r}(x) \ln(1-x) \quad (128)$$

where

$$\mu_{q,r}(x) = \binom{q+r-1}{r} \frac{(x-1)^{r+1}}{x^{r+q}}. \quad (129)$$

Substituting (125) into (112) we obtain

$$\begin{aligned}
\mathcal{S}_1(H(i), H(j), x) &= \frac{\Gamma(\tau + i')}{(1-x)^{\tau+j'}} \sum_{b=0}^{i'-1} \binom{i'-1}{b} \left(\frac{x}{1-x}\right)^b \frac{(\tau + j' + b - 1)!}{(\tau + b)!} \xi_x(\tau + j' + b - 1) \\
&= \frac{x^{i'-1} \Gamma(\tau + i')}{(1-x)^z} \sum_{b=0}^{i'-1} \binom{i'-1}{b} \left(\frac{1-x}{x}\right)^b (\tau + i' - b)_{j'-1} \xi_x(z - b - 1) \\
&= \frac{x^{i'-1} \eta_{j', i'}(\xi_x(z-1), \sqrt{x})}{(1-x)^z}. \tag{130}
\end{aligned}$$

Finally, substituting (130), (111), (110) and (106) into (100), and then combining with (98) and simplifying, we obtain the desired finite-sum expression in (27). □

#### APPENDIX IV

##### PROOF OF THEOREM 3

*Proof:* We start by following [43, 44] and applying a first-order Taylor approximation to (7) near  $\gamma = 0$  to give

$$\mathcal{I}_k \approx \log_2(e) \frac{\gamma}{N_t} \text{tr} \left( \mathbf{H}_k \mathbf{H}_k^\dagger \right). \tag{131}$$

Note that, as also mentioned in [43, 44], we emphasize that this result is only accurate for the low SNR regime; in general, requiring that the condition  $\|(\gamma/N_t) \mathbf{H}_k \mathbf{H}_k^\dagger\| < 1$  is satisfied.

Now, substituting (131) into (47) and using (57), we write the variance of the MIMO-OFDM mutual information at low SNR as follows

$$\begin{aligned}
\text{Var}^0(\mathcal{I}_{\text{ofdm}}) &= (\log_2(e))^2 \left(\frac{\gamma}{N_t}\right)^2 \left( \frac{2}{N^2} \sum_{d=1}^{N-1} (N-d) E \left[ \text{tr} \left( \mathbf{H}_0 \mathbf{H}_0^\dagger \right) \text{tr} \left( \mathbf{H}_d \mathbf{H}_d^\dagger \right) \right] \right. \\
&\quad \left. + \frac{1}{N} E \left[ \text{tr}^2 \left( \mathbf{H}_{\text{flat}} \mathbf{H}_{\text{flat}}^\dagger \right) \right] - E^2 \left[ \text{tr} \left( \mathbf{H}_{\text{flat}} \mathbf{H}_{\text{flat}}^\dagger \right) \right] \right) \tag{132}
\end{aligned}$$

where  $\mathbf{H}_{\text{flat}}$  is a flat-fading i.i.d. Rayleigh fading channel matrix. From [7], we have the following results

$$E \left[ \text{tr} \left( \mathbf{H}_{\text{flat}} \mathbf{H}_{\text{flat}}^\dagger \right) \right] = N_r N_t \tag{133}$$

$$E \left[ \text{tr}^2 \left( \mathbf{H}_{\text{flat}} \mathbf{H}_{\text{flat}}^\dagger \right) \right] = N_r N_t (1 + N_r N_t). \tag{134}$$

For the remaining expectation in (132) we write

$$\begin{aligned} E \left[ \text{tr} \left( \mathbf{H}_0 \mathbf{H}_0^\dagger \right) \text{tr} \left( \mathbf{H}_d \mathbf{H}_d^\dagger \right) \right] &= E \left[ \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} |(\mathbf{H}_0)_{i,j}|^2 \sum_{k=1}^{N_r} \sum_{\ell=1}^{N_t} |(\mathbf{H}_d)_{k,\ell}|^2 \right] \\ &= \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} E \left[ |(\mathbf{H}_0)_{i,j}|^2 |(\mathbf{H}_d)_{i,j}|^2 \right] + (N_r N_t)^2 - N_r N_t \end{aligned} \quad (135)$$

where the second line followed by noting that  $E \left[ |(\mathbf{H}_0)_{i,j}|^2 |(\mathbf{H}_d)_{k,\ell}|^2 \right] = 1$  for all  $(i, j) \neq (k, \ell)$ .

Now using (5), it can be easily shown that

$$\begin{aligned} E \left[ |(\mathbf{H}_0)_{i,j}|^2 |(\mathbf{H}_d)_{i,j}|^2 \right] &= |\rho_d|^2 E \left[ |(\mathbf{H}_d)_{i,j}|^4 \right] + (1 - |\rho_d|^2) E \left[ |(\mathbf{E})_{i,j}|^2 \right] \\ &= 1 + |\rho_d|^2. \end{aligned} \quad (136)$$

Substituting (136) into (135) we find that

$$E \left[ \text{tr} \left( \mathbf{H}_0 \mathbf{H}_0^\dagger \right) \text{tr} \left( \mathbf{H}_d \mathbf{H}_d^\dagger \right) \right] = N_r N_t (|\rho_d|^2 + N_r N_t). \quad (137)$$

The theorem now follows by substituting (137), (134), and (133) into (132) and performing some basic simplifications.

□

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TABLE I

## MIMO-OFDM PRESENCE IN STANDARDS

Standards	Technology
WLAN IEEE 802.11n	OFDM
WiMAX IEEE 802.16-2004	OFDM/OFDMA
WiMAX IEEE 802.16e	OFDMA
MBWA IEEE 802.20	OFDM
WRAN IEEE 802.22	OFDM
3GPP Release 8	OFDMA

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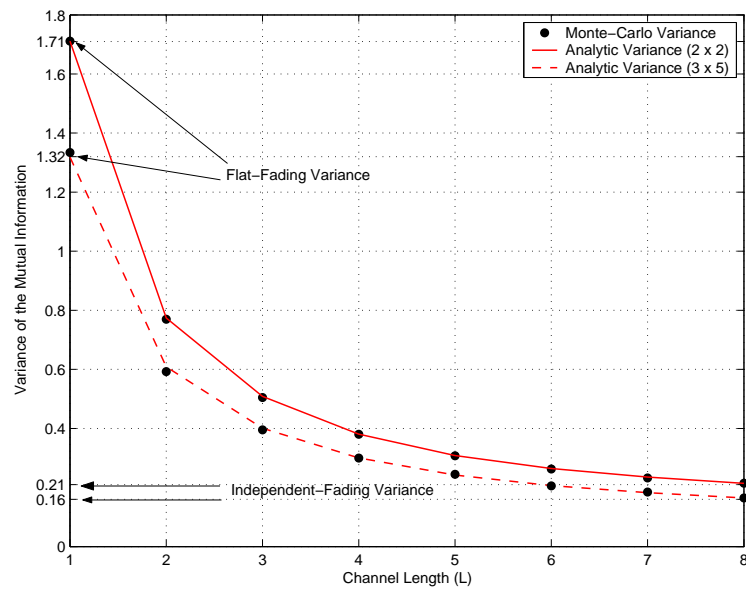


Fig. 1. Variance of the mutual information of MIMO-OFDM for different  $N_t \times N_r$  antenna configurations, and different channel lengths (uniform power delay profile). The “Analytic Variance” curves are based on (9). 8 subcarriers is considered, with SNR of 10 dB.

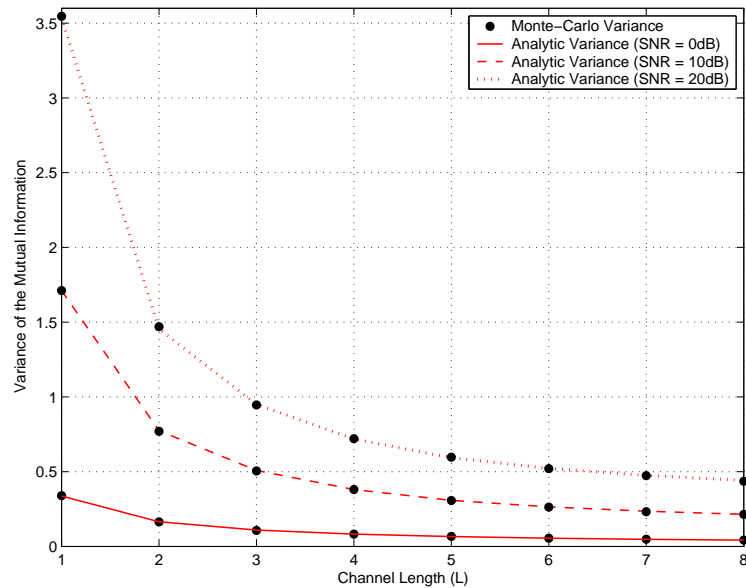


Fig. 2. Variance of the mutual information of MIMO-OFDM for different SNRs, and different channel lengths (uniform power delay profile). The “Analytic Variance” curves are based on (9). A  $2 \times 2$  system is considered with 8 subcarriers.

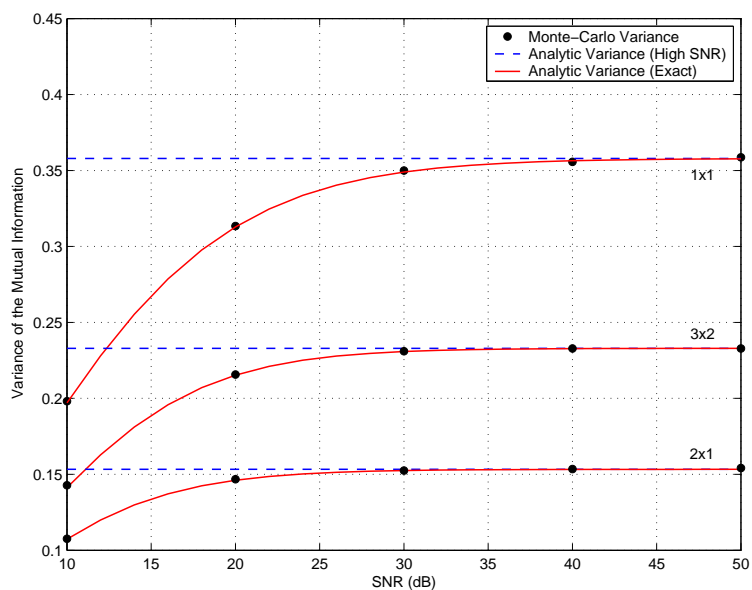


Fig. 3. Variance of the mutual information of MIMO-OFDM for different  $N_t \times N_r$  antenna configurations, and for different SNRs. The “Analytic Variance (High SNR)” lines are based on (24) for the  $2 \times 3$  case, (31) for the  $1 \times 2$  case, and (33) for the  $1 \times 1$  case. The “Analytic Variance (Exact)” curves are based on (9) for the  $2 \times 3$  case, (20) for the  $1 \times 2$  case, and (22) for the  $1 \times 1$  case. 16 subcarriers are considered, and the channel follows an 8-path uniform power delay profile.

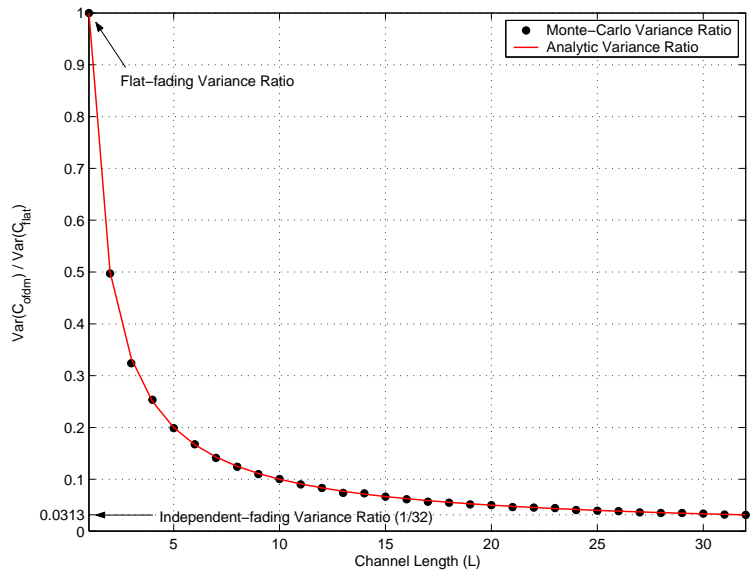


Fig. 4. Ratio of the MIMO-OFDM mutual information variance and the flat-fading MIMO mutual information variance at low SNR, for different channel lengths (uniform power delay profile). The “Analytic Variance Ratio” curve is based on (38). A  $2 \times 2$  system at  $-25$  dB is considered with 32 subcarriers.

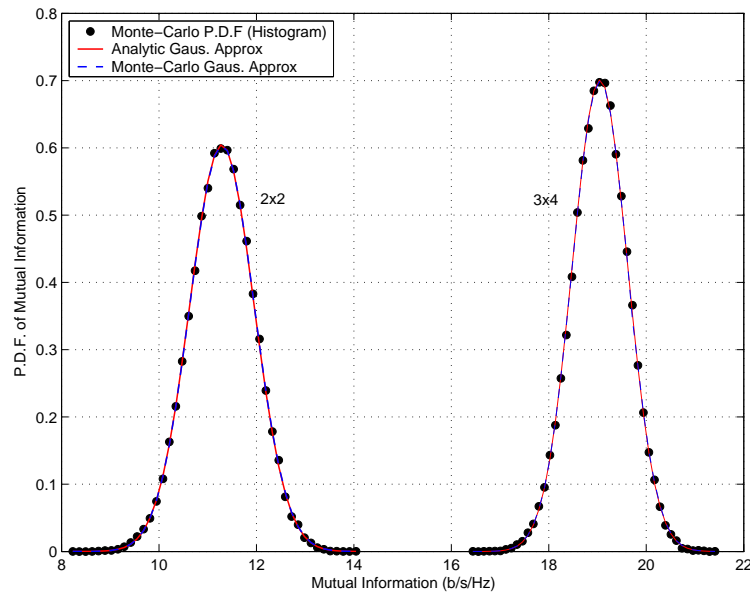


Fig. 5. P.d.f. of MIMO-OFDM mutual information for different  $N_t \times N_r$  antenna configurations. The “Analytic Gaus. Approx.” curves are based on the exact mean formula (48) and exact variance formula (9). 64 subcarriers is considered, with SNR of 20 dB. The channel follows an 8-path exponential power delay profile with  $K_{\text{exp}} = 4$ .

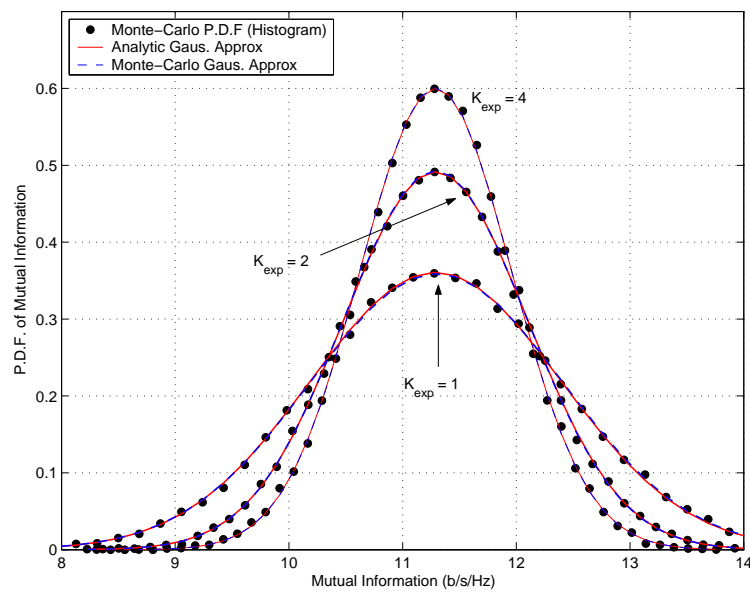


Fig. 6. P.d.f. of MIMO-OFDM mutual information for different rms delay spreads (channels follow an 8-path exponential power delay profile, with different  $K_{\text{exp}}$ ). The “Analytic Gaus. Approx.” curves are based on the exact mean formula (48) and exact variance formula (9). 16 subcarriers is considered, with SNR of 20 dB.

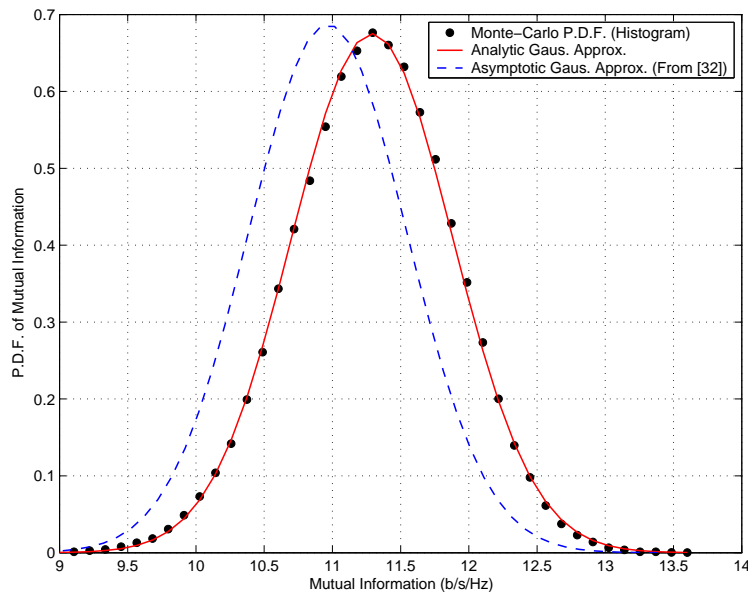


Fig. 7. P.d.f. of MIMO-OFDM mutual information. The “Analytic Gaus. Approx.” curve is based on the exact mean formula (48) and exact variance formula (9). The “Asymptotic Gaus. Approx. (From [33])” curve is based on [33, eqs. (59) and (60)].  $2 \times 2$  antennas and 32 subcarriers are considered, with 20 dB SNR. The channel follows an 8-path uniform power delay profile.

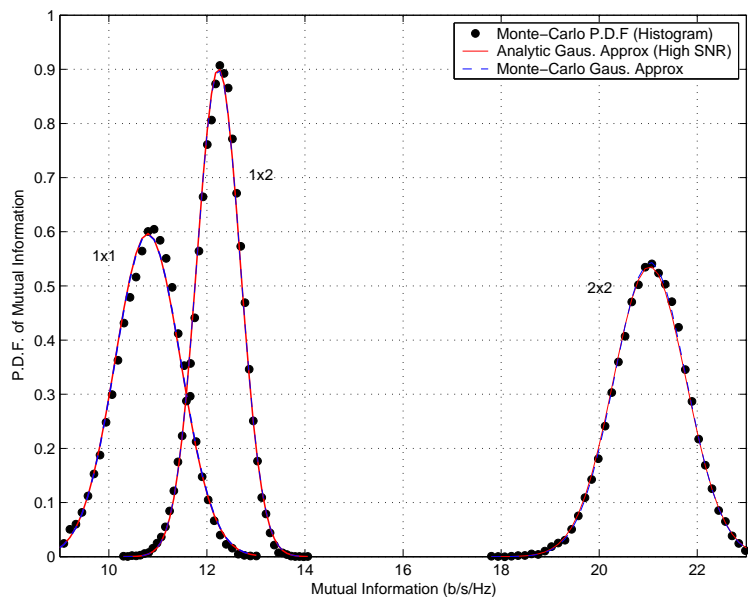


Fig. 8. P.d.f. of MIMO-OFDM mutual information at high SNR. The “Analytic Gaus. Approx (High SNR)” curves are based on the high SNR mean formula [3, Theorem 2], and the variance formula (24) for the  $2 \times 2$  case, (31) for the  $1 \times 2$  case, and (33) for the  $1 \times 1$  case. 16 subcarriers is considered, with SNR of 35 dB. The channel follows an 8-path exponential power delay profile with  $K_{\text{exp}} = 4$ .

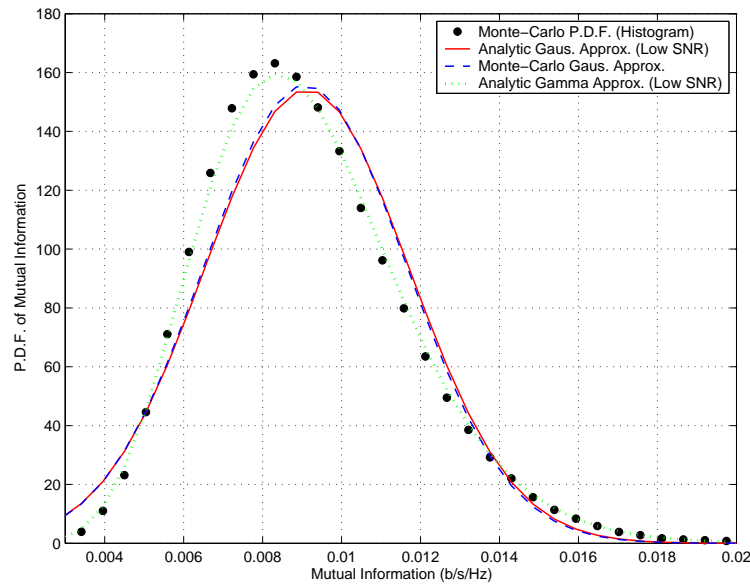


Fig. 9. P.d.f. of MIMO-OFDM mutual information at low SNR. The “Analytic Gaus. Approx (Low SNR)” and “Analytic Gamma Approx (Low SNR)” curves are based on the low SNR mean formula obtained by combining (131) and (133), and the low SNR variance formula (34). A  $2 \times 2$  system with 32 subcarriers is considered, with SNR of  $-25$  dB. The channel follows a 4-path exponential power delay profile with  $K_{\text{exp}} = 2$ .

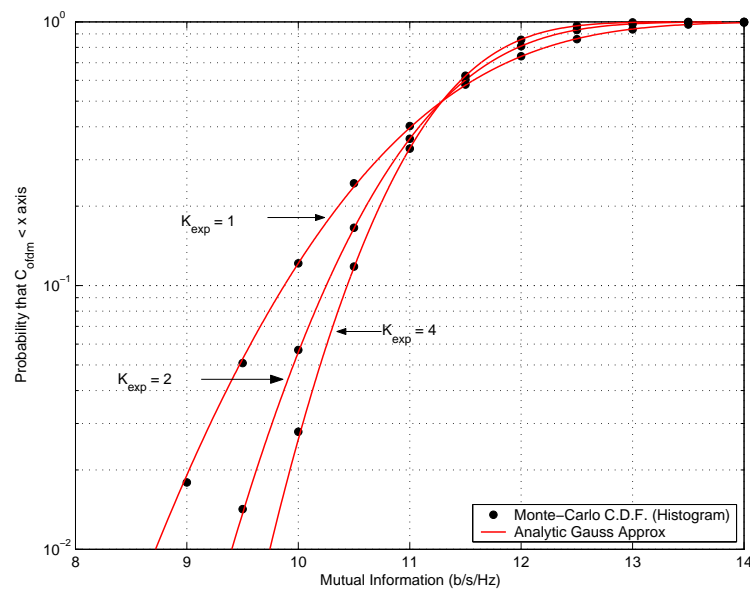


Fig. 10. C.d.f. of MIMO-OFDM mutual information for different rms delay spreads (channels follow an 8-path exponential power delay profile, with different  $K_{\text{exp}}$ ). The “Analytic Gaus. Approx” curves are based on the exact mean formula (48) and exact variance formula (9). A  $2 \times 2$  system with 16 subcarriers is considered, with SNR of 20 dB.