

QAM and PSK Codebooks for Limited Feedback MIMO Beamforming

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Abstract

This paper considers the problem of beamforming in multiple-input multiple-output (MIMO) wireless systems. Assuming perfect channel state information at the receiver, the choice of the beamforming vector is made possible through a noiseless limited-rate feedback to the transmitter. This paper proposes the use of beamforming codebooks based on quadrature amplitude modulation (QAM) and phase-shift keying (PSK) constellations, which essentially eliminates the need for storage of the codebook. We show that such codebooks perform arbitrarily close to the perfect feedback case as the constellation size increases, and that full diversity order is achieved. We demonstrate an equivalence between the beamforming codebook search problem with that of noncoherent sequence detection. Based on this we propose fast beamforming vector search algorithms. Monte-Carlo simulations are presented to show that the performance is comparable to the best known codebooks, and that the search complexity can be reduced by several orders of magnitude.

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I. INTRODUCTION

Multiple-input multiple-output (MIMO) systems can provide increased reliability in wireless communication links by exploiting the spatial diversity due to the increased number of transmit-receive paths. A simple technique to obtain the highest possible diversity order is to employ transmit beamforming and receive combining, which simultaneously improves the array gain. This technique requires that the transmitter has channel state information in the form of a transmit beamforming vector. It is often impractical to have a reciprocal channel for the transmitter to estimate the channel, and thus a small number of bits are sent via a feedback path for the transmitter to recreate the beamforming vector. Such systems are known as limited feedback systems (see [1] and references therein).

In these limited feedback systems, the transmitter and receiver share a codebook of possible beamforming vectors indexed by a number of bits. The receiver chooses a beamforming vector from the codebook on the basis of maximizing the effective signal-to-noise ratio (SNR) after combining, and sends the corresponding bits to the transmitter. We examine two beamforming codebook strategies in this paper. Firstly, we consider maximum ratio transmission (MRT), where the beamforming vectors are constrained to have unit length, so that the energy expended in each packet transmission is unchanged [2, 3]. Secondly, we consider equal gain transmission (EGT), where the transmit power of each antenna is unaffected, and thus the amplifier requirements are not increased [4].

The most straightforward approach to designing a limited feedback system is to employ scalar quantization where each component of the beamforming vector is quantized separately. More advanced approaches to the limited feedback design problem involve designing beamforming vector codebooks using the minimum number of feedback bits possible for a given effective SNR after combining [3], neglecting the search and storage requirements. The codebook design strategies which have been suggested use either numerical optimization techniques [3–7], or for larger systems the codebooks can be randomly generated, *i.e.* random vector quantization (RVQ) [8]. Such random codebooks have been shown to be asymptotically optimal as the number of bits and antennas increase [9, 10].

Unfortunately, the codebook size increases exponentially with the number of transmit antennas to maintain a given effective SNR or capacity loss with respect to the ideal unquantized system [3, 7, 11]. Since the codebooks have no structure an exhaustive search is usually required. For time-varying channels, the resulting delay due to the excessive search time reduces the effectiveness of the beamforming vector when employed at the transmitter [12]. Non-exhaustive methods for searching unstructured codebooks at the expense of increased memory requirements have been well documented in [13]. One of these methods is a tree-search [5, 14], where storage of the tree and codebook is required. An additional consequence of exponential growth in codebook size is that even storage of the codebook may be infeasible for large numbers of antennas.

In this paper, we propose schemes to reduce both the search and storage complexity. This is enabled by using structured codebooks where each vector in the codebook is a sequence of symbols from an M -ary quadrature amplitude modulation (QAM) or M -ary phase-shift keying (PSK) constellation. The QAM codebooks are used for quantizing the ideal infinite-precision MRT vector, and since PSK symbols have equal envelope, the PSK codebooks are used for quantized EGT. Before application at the transmitter, the vectors are scaled to have unit norm so as to satisfy the power constraints.

Since QAM and PSK constellations have simple bit-to-symbol mapping algorithms no codebook storage is required at either the transmitter or receiver¹. We show that the performance can be arbitrarily close to the ideal unquantized system as $M \rightarrow \infty$ and that the codebooks achieve the full diversity order. We also discuss the improved peak-to-average power ratio (PAPR) obtained, as compared to using unstructured codebooks.

Our primary motivation for proposing the use of QAM and PSK codebooks is to exploit their structure to reduce the codebook search complexity. To do this, we first demonstrate the equivalence between the problem of finding the optimal beamforming vector in the codebook and the problem of noncoherent sequence detection. We then provide optimal codebook search algorithms based on the fast noncoherent sequence detection algorithms given in [15–17]. We also show how further reductions in complexity may be obtained, while maintaining near identical performance, by using the suboptimal noncoherent sequence detection algorithm from [18]. We show that the complexity of the new algorithm can be orders of magnitude smaller than an exhaustive search. We also show that even lower complexity scalar quantization schemes can be developed with a compromise on performance.

Importantly, we show that these reduced-search algorithms do not reduce the diversity order of the system. We also calculate the average effective channel gain and provide an upper bound on the capacity of the beamforming system employing PSK-based codebooks and scalar quantization.

Finally, we compare via simulation the QAM and PSK-based schemes to the codebooks obtained via numerical optimization in [3, 7, 19, 20], available for small numbers of antennas and codebook sizes, and for large number of antennas we compare to randomly generated codebooks. We show that the performance is similar for the same number of feedback bits, and much improved for a given computational complexity. We also show that the performance of the proposed quantization algorithms significantly improves over a scalar quantization approach.

II. MIMO BEAMFORMING SYSTEM MODEL

We consider a single-user, narrowband wireless MIMO system employing transmit beamforming and receive combining with N_T transmit antennas and N_R receive antennas (for more details see [3]). The channel $\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$

¹The index bits need not be relayed in the same modulation format.

is assumed to be independent and identically distributed (i.i.d.) complex Gaussian with unit variance. The channel state information (CSI) is perfectly known at the receiver.

Given the transmitted symbol $x \in \mathbb{C}$, the received symbol $y \in \mathbb{C}$ is given by

$$y = \mathbf{z}^\dagger \mathbf{H} \mathbf{w} x + \mathbf{z}^\dagger \mathbf{n} \quad (1)$$

where $\mathbf{w} \in \mathbb{C}^{N_T \times 1}$ and $\mathbf{z} \in \mathbb{C}^{N_R \times 1}$ are the beamforming and combining vectors respectively with $\|\mathbf{w}\|, \|\mathbf{z}\| = 1$. The vector $\mathbf{n} \in \mathbb{C}^{N_R}$ consists of i.i.d. circularly symmetric complex Gaussian variables, each with variance N_0 . The average symbol energy is $E_x = E[|x|^2]$. The instantaneous signal-to-noise ratio (SNR), ρ , is

$$\rho \triangleq \frac{E_x \Gamma(\mathbf{H})}{N_0}$$

where $\Gamma(\mathbf{H}) \triangleq |\mathbf{z}^\dagger \mathbf{H} \mathbf{w}|^2$ is the effective channel gain.

Note that given any \mathbf{w} , it is well known that the combining vector which maximizes the SNR is given by maximum ratio combining (MRC) where $\mathbf{z} = \mathbf{H} \mathbf{w} / \|\mathbf{H} \mathbf{w}\|$. The resulting effective channel gain is $\Gamma_{\text{MRC}}(\mathbf{H}) = \|\mathbf{H} \mathbf{w}\|^2$.

We consider the case where there is a low-rate, error-free, zero-delay feedback channel from the receiver to the transmitter, as in [1, 3]. The receiver and transmitter share a codebook \mathcal{C} of N possible beamforming vectors of unit magnitude, which are indexed by $B = \lceil \log_2 N \rceil$ bits.

To maximize the SNR, the receiver chooses the beamforming vector from the codebook according to

$$\mathbf{w} = \arg \max_{\mathbf{v} \in \mathcal{C}} \frac{\|\mathbf{H} \mathbf{v}\|^2}{\|\mathbf{v}\|^2}. \quad (2)$$

and then sends the corresponding index bits to the transmitter.

The beamforming scheme employing the ideal unquantized beamforming vector with $\|\mathbf{w}\| = 1$ is known as maximum ratio transmission (MRT) [2]. In this case, the ideal infinite-precision beamforming vector is given by the right-singular vector of \mathbf{H} corresponding to the largest singular value of \mathbf{H} . We denote this vector \mathbf{w}^{opt} . The effective channel gain when using MRT and MRC is denoted $\Gamma_{\text{MRT,MRC}}(\mathbf{H})$.

For equal gain transmission (EGT), the beamforming vector has the property that $|w_t| = 1/\sqrt{N_T}$ for all t . The effective channel gain when using EGT and MRC is denoted $\Gamma_{\text{EGT,MRC}}(\mathbf{H})$. In this paper, we generate EGT beamforming vectors by quantizing the optimal MRT vector \mathbf{w}^{opt} so as to maximize the effective SNR under the per-antenna power constraint.

III. CONSTELLATION BASED CODEBOOK DESIGN

In this section, we propose a method of beamforming codebook construction using sequences of symbols drawn from an M -ary QAM or PSK constellation. We then show how the beamforming vectors are indexed efficiently.

A. QAM Codebook Construction

For QAM, the proposed codebook \mathcal{C} is the set of all M^{N_T} possible sequences of M -ary QAM symbols of length N_T , with some sequences that we will show to be redundant removed. In this paper we focus on some specific QAM constellations. The QAM constellations we consider satisfy the following properties: 1) the real and imaginary part of each constellation point is an odd integer; and 2) the constellation exhibits 2-fold rotational symmetry² on the complex plane about the origin, so that there exists at least one pair of antipodal constellation points $u, -u$. Specifically, we consider square QAM with $M = 4, 16, 64$ and rectangular 8-QAM where the constellation points have real part $\pm 1, \pm 3$ and imaginary part ± 1 . However, it is not hard to extend the algorithms and analysis to other constellation shapes such as cross or circular.

Each QAM vector is a basis for a subspace in \mathbb{C}^{N_T} . Moreover, two vectors $\mathbf{v}_1, \mathbf{v}_2$ are a basis for the same subspace in \mathbb{C}^{N_T} , if there exists some $\gamma \in \mathbb{C}$ such that $\mathbf{v}_1 = \gamma \mathbf{v}_2$, implying $\|\mathbf{H}\mathbf{v}_1\| / \|\mathbf{v}_1\| = \|\mathbf{H}\mathbf{v}_2\| / \|\mathbf{v}_2\|$. There exist sets of QAM vectors which share the same subspace and always give the same effective channel gain regardless of \mathbf{H} . Therefore the redundant vectors can be removed to reduce the number of feedback bits. When $|\gamma| = 1$, this is a phase ambiguity. Since square QAM constellations have 4-fold rotational symmetry, each QAM vector has at least three other QAM vectors in the codebook that describe the same subspace. Hence, only a quarter of all possible M^{N_T} constellations are required, and by removing redundant sequences this saves two bits in the feedback path. To remove this ambiguity, we apply the restriction that $\text{Re}\{v_1\} \geq 0$ and $\text{Im}\{v_1\} > 0$, for all $\mathbf{v} \in \mathcal{C}$. Thus, two less bits are required to describe v_1 and an upper bound on the number of bits required for the codebook is $B \leq \lceil N_T \log_2 M - 2 \rceil$. The version of 8-QAM considered in this paper exhibits only 2-fold rotational symmetry, hence we apply the restriction that $\text{Im}\{v_1\} > 0$ and it follows that $B = 3N_T - 1$.

Another form of ambiguity arises between QAM vectors when $|\gamma| \neq 1$, which we call a divisor ambiguity [21]. Closed-form expressions for calculating the number of redundant sequences due to divisor ambiguity were provided in [21] using number theoretic techniques in the context of noncoherent sequence detection. For the constellations examined in this paper, the number of such sequences is not significant enough to reduce the required bits in the feedback path by 1, regardless of the number of antennas. Hence, since we are considering only a single channel, the upper bounds given in the previous paragraph are treated as equalities in the rest of this paper. However, small reductions in the feedback rate may be obtained for some large M when quantizing over a number of channels, such as in an OFDM system.

Note that as M increases there exist candidate vectors in the QAM lattice that give an effective channel gain

²An object exhibits n -fold rotational symmetry in 2 dimensions if a rotation of $2\pi/n$ does not change the object.

$\|\mathbf{H}\mathbf{w}\|^2$ arbitrarily close to the optimum $\|\mathbf{H}\mathbf{w}^{\text{opt}}\|^2$.

B. PSK Codebook Construction

For PSK-based codebooks, the codebook \mathcal{C} is formed by normalizing the set of all M^{N_T} possible sequences of M -ary PSK symbols of length N_T to have unit magnitude. Due to the π/M rotational symmetry of PSK constellations, each PSK sequence has $M - 1$ other PSK sequences that describe the same subspace. To remove this ambiguity, we apply the restriction that $w_1 = 1/N_T$. Thus, no bits are required to describe w_1 and the number of bits required for the codebook is $B \leq \lceil (N_T - 1) \log_2 M \rceil$.

As M increases there exist candidate vectors in the PSK-based codebook that give an effective channel gain arbitrarily close to that obtained by using the ideal unquantized EGT vector.

IV. CODEBOOK PROPERTIES

We now discuss the properties of the codebook which suggest that the performance of the QAM and PSK codebooks will be comparable to the unstructured codebooks. We show that the codebooks achieve the full diversity order and have a better PAPR than unstructured MRT codebooks.

A. Diversity order

A system achieves diversity order D if the probability of symbol error, P_e , averaged over \mathbf{H} satisfies

$$\lim_{E_x/N_0 \rightarrow \infty} \frac{\log P_e(E_x/N_0)}{\log(E_x/N_0)} = -D. \quad (3)$$

For an $N_T \times N_R$ system the maximum achievable diversity order is $D = N_T N_R$ [2]. To show that QAM and PSK-based codebooks achieve the full diversity order, we first note the following result from [22, Theorem1].

Theorem 1 (Love and Heath [22]): A wireless system employing beamforming and combining over memoryless, correlated Rayleigh fading channels provides full diversity order if and only if the vectors in the beamforming codebook span \mathbb{C}^{N_T} and there exists a set of valid combining vectors that span \mathbb{C}^{N_R} .

Note that the theorem includes i.i.d. Rayleigh fading MIMO channels as a special case. This result is used in the proof of the following lemma.

Lemma 1: A wireless system employing a QAM or PSK-based beamforming codebook with receiver combining over memoryless, correlated Rayleigh fading channels provides full diversity order if the set of valid combining vectors span \mathbb{C}^{N_R} .

Proof: The proof proceeds by first constructing a set of vectors spanning \mathbb{C}^{N_T} so that Theorem 1 is satisfied. To do this we can construct a matrix with linearly independent rows, which are the required spanning vectors, and

represent valid codewords. Consider the matrix $\mathbf{B} = \mathbf{1} + (e^{\frac{j2\pi}{M}} - 1)\mathbf{I}$, where $\mathbf{1}$ is an $N_T \times N_T$ matrix with all entries equal to one, and \mathbf{I} is the $N_T \times N_T$ identity matrix. It can be seen that the row vectors of \mathbf{B} are valid codewords³ from an M -ary PSK-based codebook. Noting that $\det(\mathbf{B}) \neq 0$ if and only if \mathbf{B} spans \mathbb{C}^{N_T} , we have

$$\begin{aligned} \det(\mathbf{B}) &= (e^{\frac{j2\pi}{M}} - 1)^{N_T} \det(\mathbf{I} + (e^{\frac{j2\pi}{M}} - 1)^{-1}\mathbf{1}) \\ &= (e^{\frac{j2\pi}{M}} - 1)^{N_T} \left(1 + \frac{N_T}{e^{\frac{j2\pi}{M}} - 1}\right) \\ &= (e^{\frac{j2\pi}{M}} - 1)^{N_T-1} (e^{\frac{j2\pi}{M}} - 1 + N_T). \end{aligned} \quad (4)$$

Now, (4) is equal to zero only if $M = N_T = 2$. In this case, we can use rows of the linearly independent matrix $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ as codewords. Therefore a set of beamforming vectors that span \mathbb{C}^{N_T} can be constructed for any value of N_T from an M -ary PSK codebook. It follows from Theorem 1 and the assumption that the combining vectors span \mathbb{C}^{N_R} , that PSK-based codebooks achieve the full diversity order. Since antipodal BPSK sequences are a subset of any QAM-based codebook, it follows that QAM-based codebooks achieve the full diversity order. ■

Note that the proof could be simplified if we did not consider M -ary PSK for all M by noting that BPSK sequences are subsets of QAM and M -ary PSK codebooks with even M .

B. Peak-to-Average Power Ratio

The PAPR is an important issue in codebook design, as it can greatly affect the hardware requirements of multiple-antenna systems.

We consider the peak-to-average power ratio (PAPR) taken to be the ratio between the maximum possible power scaling performed at any one antenna, over all possible channels, and that of the average power of the antenna. We assume that there is no additional variable power scaling performed at the transmitter (e.g. due to automatic gain control).

Firstly, note that the average transmit power per antenna, P_{ave} is $\frac{1}{N_T}$ since $\|\mathbf{w}\| = 1$. We now turn to calculating the peak power at any one antenna, defined as $P_{\text{peak}} = \max_{\mathbf{w} \in \mathcal{C}} \max_t |w_t|^2$.

For unstructured codebooks, if the only design constraint applied is that $\|\mathbf{w}\| = 1$, it is possible that there exists a codebook vector corresponding to all the power being concentrated in one transmit antenna. It follows that P_{peak} can be as high as one resulting in a worst-case PAPR of $\text{PAPR} = N_T$.

For QAM based codebooks, we have by construction that each antenna has a non-zero power allocation, and hence the PAPR is always strictly less than N_T . Furthermore, the PAPR is also bounded by the constellation size.

³As a consequence of the bit saving restriction that $w_1 > 0$ discussed in Section III-B, the first row vector must be rotated to be a valid beamforming vector. This does not affect the rank of the matrix.

For example, for M -ary square QAM constellations, the peak transmit power at any antenna due to beamforming is due to the case where one antenna is designated one of the four outermost QAM constellation points *e.g.* $(\sqrt{M} - 1) + j(\sqrt{M} - 1)$, and the rest are designated one of the four innermost QAM constellation points *e.g.* $1 + j$. Remembering that the beamforming vector is scaled to have unit magnitude, the maximum magnitude of any element of the beamforming vector is therefore

$$P_{\text{peak}} = \frac{(\sqrt{M} - 1)^2}{(N_T - 1) + (\sqrt{M} - 1)^2}.$$

Hence

$$\text{PAPR} = \frac{N_T(\sqrt{M} - 1)^2}{(N_T - 1) + (\sqrt{M} - 1)^2} \leq (\sqrt{M} - 1)^2$$

since $\sqrt{M} \geq 2$ and hence $(\sqrt{M} - 1)^2 \geq 1$. Hence, employing QAM-based codebooks instead of naively designed unstructured quantized MRT codebooks implies that amplifiers with smaller dynamic range can be used, reducing the expense of the system. Alternatively, the power backoff required to avoid nonlinear amplifier effects may be reduced, increasing the range or throughput of the system.

V. CODEBOOK SEARCH

In this section we show how the codebook search over the proposed QAM and PSK-based codebooks can be performed optimally with low complexity by observing an equivalence with noncoherent sequence detection, and then applying fast noncoherent algorithms developed for that problem.

We also consider a suboptimal approach for the QAM codebook search, which reduces the complexity considerably without any noticeable loss in performance. We also compare scalar quantization approaches, which have low complexity but suffer a performance loss. We provide a unified framework for the algorithms to clarify the complexity-performance tradeoff of the schemes.

A. Codebook Search via Singular Vector Quantization

We first show that for the case where $N_R \leq 2$, the search in (2) is equivalent to finding the closest beamforming vector *in angle* to the optimal unquantized beamforming vector. For the case $N_R > 2$, we show that it is also a good first-order approximation. We denote this approach as singular vector quantization (SVQ), because it effectively performs a codebook search by quantizing the righthand singular vector associated with the largest singular value of the channel matrix.

We start by calculating the effective channel gain, in terms of angles, for a particular beamforming vector \mathbf{v} . Using MRC we have

$$\Gamma_{\text{MRC}}(\mathbf{H}) = \|\mathbf{H}\mathbf{v}\|^2 = \sum_{k=1}^K \sigma_k^2 |\mathbf{v}^\dagger \mathbf{u}_k|^2$$

where $K = \text{rank}(\mathbf{H}) > 0$, σ_k^2 is the k th ordered singular value of \mathbf{H} , and \mathbf{u}_k is the corresponding k th right singular vector. Note that $\mathbf{u}_1 = \mathbf{w}^{\text{opt}}$ by definition. Now defining

$$\cos^2 \theta(\mathbf{v}, \mathbf{u}) \triangleq \frac{|\mathbf{v}^\dagger \mathbf{u}|^2}{\|\mathbf{v}\|^2 \|\mathbf{u}\|^2}$$

where by definition $\theta(\mathbf{v}, \mathbf{u})$ is the principal angle between the subspaces $\mathbf{v}\mathbb{C}$ and $\mathbf{u}\mathbb{C}$, and recalling that $\|\mathbf{v}\| = \|\mathbf{u}_k\| = 1$ we have

$$\Gamma_{\text{MRC}}(\mathbf{H}) = \sum_{t=1}^K \sigma_t^2 \cos^2 \theta(\mathbf{v}, \mathbf{u}_t). \quad (5)$$

For the MISO case, or for the MIMO case where $\text{rank}(\mathbf{H}) = 1$, there is only one singular vector and hence $\Gamma_{\text{MRC}}(\mathbf{H}) = \sigma_1^2 \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}})$. Therefore the maximization in (2) can be achieved by searching for the closest vector in \mathcal{C} in angle to \mathbf{w}^{opt} , *i.e.*

$$\mathbf{w} = \arg \max_{\mathbf{v} \in \mathcal{C}} \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}). \quad (6)$$

For the MIMO case where $\text{rank}(\mathbf{H}) > 1$, we obtain the upper bound of (5) as follows,

$$\Gamma_{\text{MRC}}(\mathbf{H}) \leq \sigma_1^2 \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}) + \sigma_2^2 \sum_{k=2}^K \cos^2 \theta(\mathbf{v}, \mathbf{u}_k) = \sigma_1^2 \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}) + \sigma_2^2 \sin^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}})$$

where we have used the fact that

$$\sum_{k=1}^K \cos^2 \theta(\mathbf{v}, \mathbf{u}_k) = \sum_{k=1}^K |\mathbf{v}^\dagger \mathbf{u}_k|^2 = 1.$$

Similarly, we obtain the lower bound

$$\Gamma_{\text{MRC}}(\mathbf{H}) \geq \sigma_1^2 \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}) + \sigma_K^2 \sin^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}).$$

The upper and lower bounds are equal for the cases $N_R = 1, 2$. Thus for $N_R \leq 2$, finding the beamforming vector closest in angle to \mathbf{w}^{opt} maximizes the effective channel gain.

Also note that the upper and lower bounds both approach $\sigma_1^2 \cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}})$ as $\cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}}) \rightarrow 1$ (*i.e.* for beamforming vectors in the codebook close to the optimal). Moreover, note that for $N_R > 2$, $\|\mathbf{H}\mathbf{v}\| / \|\mathbf{v}\|$ and $\cos^2 \theta(\mathbf{v}, \mathbf{w}^{\text{opt}})$ are still maximized by the same unquantized vectors. Therefore, we are motivated to use codebook search algorithms that limit the search to only those vectors close in angle to \mathbf{w}^{opt} , regardless of the number of receive antennas.

In summary, we see that the SVQ approach of calculating the singular vector \mathbf{w}^{opt} , and searching the beamforming vector codebook according to (6) gives optimal or near-optimal performance, depending on the scenario. For systems

with $N_R \leq 2$ searching according to (6) is optimal. For $N_R > 2$, searching according to (6) is expected to at least give a good approximation, if not the optimal. The approximation comes about in this case since there is no guarantee that the optimum beamforming vector, according to (5), is in the search space. Finally, we note that the SVQ approach can also be used in the limited feedback MIMO broadcast scenario [23].

B. Equivalence between SVQ and Noncoherent Sequence Detection

In this section we show that there exists an equivalence relationship between SVQ using the angular metric and the problem of sequence detection over unknown deterministic flat-fading channels, e.g. [21, 24]. The equivalence can be seen by noting that the cost function in (6) is equivalent to noncoherent detection in the form of the generalized likelihood ratio test (GLRT) [25]. Specifically, consider the detection of a discrete valued input $\mathbf{x} \in \mathbb{C}^T$, given an output

$$\mathbf{y} = h\mathbf{x} + \mathbf{n} \quad (7)$$

where $\mathbf{n} \in \mathbb{C}^T$ is a vector of i.i.d. additive white Gaussian noise and $h \in \mathbb{C}$ is an unknown deterministic channel assumed constant for a period of T symbols. As was shown in [24], the GLRT-optimal data estimate $\hat{\mathbf{x}}^{\text{GLRT}}$ is obtained from the received data by solving

$$\hat{\mathbf{x}}^{\text{GLRT}} = \arg \min_{\hat{\mathbf{x}}} \min_{\hat{h}} \|\mathbf{y} - \hat{h}\hat{\mathbf{x}}\|^2 = \arg \max_{\hat{\mathbf{x}}} \cos^2 \theta(\mathbf{y}, \hat{\mathbf{x}}). \quad (8)$$

Note also that the angular metric $\cos^2 \theta(\mathbf{y}, \hat{\mathbf{x}})$ is also the metric for maximum likelihood (ML) estimation of a sequence of PSK signals over an uniformly distributed phase noncoherent AWGN channel [26], *i.e.*,

$$\hat{\mathbf{x}}^{\text{ML}} = \arg \max_{\hat{\mathbf{x}}} E_{\theta} [p(\mathbf{y}|\hat{\mathbf{x}}, \theta)] = \arg \max_{\hat{\mathbf{x}}} |\mathbf{y}^{\dagger} \hat{\mathbf{x}}|^2 = \arg \max_{\hat{\mathbf{x}}} \cos^2 \theta(\mathbf{y}, \hat{\mathbf{x}}).$$

The important step in this paper is to recognize that the detection task of searching the transmitted symbols in (8) is equivalent to the beamforming codebook search in (6). It follows that the algorithms developed for the detection problem can then be applied to the beamforming vector codebook search problem.

For sequences of T symbols drawn in an i.i.d manner from a specific constellation, it was shown in [27] that the GLRT-optimal data sequence estimate can be found in time polynomial with T . An explicit lattice decoding algorithm for QAM symbol detection over fading channels that gives the GLRT-optimal estimate with complexity $O(T^3)$ was given in [15]. For the constellations BPSK and QPSK, ML noncoherent PSK sequence detection can be performed using an algorithm with complexity $O(T \log T)$ [16, 17]. Thus we are able to use the algorithms in [15–18] as a basis for a low-complexity algorithm to find the optimal beamforming vector according to (2), where of course the sequences are of length N_T .

Once the equivalence is made, the algorithm manifests itself directly in terms of a ‘black-box’ implementation of the noncoherent sequence detection algorithms, which perform the SVQ operation. We first discuss the black-box implementation of the algorithms, before providing a unified framework of their operation which will explain the relative performance and complexity advantages of using these algorithms. Table I gives pseudo-code for the proposed codebook search for both QAM and PSK-based codebooks.

The function QAM-SVQ(\mathbf{w}^{opt}) on line 4 performs a quantization of \mathbf{w}^{opt} by choosing a vector from a QAM-based codebook. We propose three different methods for the codebook search each achieving a different complexity/performance tradeoff.

- **QAM-SVQ:** The codebook search is performed using the GLRT-optimal noncoherent sequence detection algorithm in [15] which searches in a reduced space but still guarantees that the QAM codeword closest in angle to the input vector is found. This algorithm’s complexity is vastly reduced compared to an exhaustive search, as only $O(N_T^2)$ vectors in the codebook are tested. When $N_R > 2$, we recall that the closest-in-angle search is not strictly optimal, so we improve the performance by making the modification to the algorithm that each codeword examined is tested according to the effective channel gain $\|\mathbf{H}\mathbf{v}\|^2 / \|\mathbf{v}\|^2$ instead of the angle.
- **Suboptimal QAM-SVQ:** This codebook search is a suboptimal simplification of the QAM-SVQ algorithm, using the suboptimal detection algorithm in [18, Section VI.B]. This algorithm examines far fewer codewords than QAM-SVQ, but the beamforming vector closest in angle to \mathbf{w}^{opt} is nearly always found. We will see via simulation in Section VII that there is negligible performance loss relative to QAM-SVQ.
- **Scalar QAM-SVQ:** In this codebook search, the ideal beamforming vector \mathbf{w}^{opt} is multiplied by a scalar value and then each element is quantized with a scalar Euclidean quantizer. This has the lowest complexity but comes with a performance loss.

After the quantization, the function Rotate-To-First-Quadrant(\mathbf{w}) on line 5 of Table I is invoked which multiplies the vector \mathbf{w} by a complex unit $\pm 1, \pm i$ so that for square constellations $0 \leq \arg w_1 < \frac{\pi}{2}$, or for rectangular constellations $0 \leq \arg w_1 < \pi$. This gives a valid beamforming vector from the codebook. The function QAM-to-Bits(\mathbf{w}) performs a QAM bit demapping and removes the first one or two bits of the sequence that are redundant, due to the phase ambiguity, to provide the feedback bit sequence.

Similarly, for the PSK-based codebooks the function PSK-SVQ(\mathbf{w}^{opt}) on line 7 performs a quantization of \mathbf{w}^{opt} by choosing a vector from a PSK-based codebook using one of the two following methods.

- **PSK-SVQ:** The codebook search is performed based on the detection algorithm from [16, 17] which provides the closest PSK vector in angle to the input vector. As for the QAM case, the input is the ideal unquantized

MRT beamforming vector \mathbf{w}^{opt} .

- **Scalar-PSK:** This codebook search is similar to that described for scalar QAM-SVQ. The performance is the easiest to analyze, and we will see via simulation this provides a tight lower bound on PSK-SVQ performance for moderate number of feedback bits per antenna.

After the PSK quantization, the function Rotate-To-Real-Axis(\mathbf{w}) on line 8 of Table I rotates the vector \mathbf{w} by a complex number so that $\text{Im}\{w_1\} = 0$ to give a valid beamforming vector. The function PSK-to-Bits(\mathbf{w}) performs a PSK bit demapping ignoring the first symbol.

For both QAM and PSK-based codebooks, the selected index bits are sent to the transmitter, where the bits are remapped to the constellation points to create the beamforming vector \mathbf{w} , which is then scaled so that $\|\mathbf{w}\| = 1$.

In Section VII, we will see via simulation that the algorithms which search the codebook for the codeword closest in angle to the ideal unquantized beamforming vector produce very good results.

We now discuss the algorithms in more detail so as to highlight the differences in the complexity and performance gains.

C. Unified Framework of Codebook Search Algorithms

In the previous section, a number of search algorithms were referred to, from [15–18]. In this subsection, we provide a unified overview of the operation of each of these search algorithms. We discuss their relative complexities, in terms of N_C , the number of codewords that are enumerated and have their metric calculated by each algorithm. This allows us to show the merit of the suboptimal QAM-SVQ approach and the benefit of the “closest-in-angle” based search algorithms compared to a simple scalar quantization.

Recall from (7) the relationship between the SVQ angular cost function and the Euclidean cost function. More generally, for both QAM and PSK each of the SVQ search algorithms discussed in the previous section can be formulated as an optimization of the following form

$$\mathbf{w} = \arg \min_{\mathbf{v} \in \mathcal{C}} \min_{\beta \in \mathcal{R}} \|\mathbf{w}^{\text{opt}} - \beta \mathbf{v}\|^2 \quad (9)$$

where $\mathcal{R} \subseteq \mathbb{C}$ is the search region of the algorithm, which is different for each algorithm and whether QAM or PSK is used. In the following subsections we describe the different search regions for each algorithm, and outline the search procedure.

Note that from a geometrical perspective it is useful to think of β as a scaling parameter, that allows any scaled version of a codeword to be tested in terms of Euclidean distance. Certainly, the scaled version of the codeword that minimizes (9) will also minimize (6).

1) *QAM-SVQ*: For QAM-SVQ, the proposed search algorithm from [15] performs a search, which when viewed in terms of (9) corresponds to the case where \mathcal{R} is the entire complex plane \mathbb{C} . In essence, the search algorithm limits the search space by focusing on the complex plane defined by $\beta^{-1}\mathbf{w}^{\text{opt}}$, and searching only the vectors in \mathcal{C} that are near the plane. More specifically, it only searches vectors which have a nearest neighbor region which cuts through the plane. The nearest neighbor region of a codeword \mathbf{x} is defined as the region in \mathbb{C}^{N_T} for which \mathbf{x} is the closest codeword in terms of Euclidean distance. In Figure 1(a) we demonstrate the plane with axes $\text{Re}\{\beta^{-1}\}$ and $\text{Im}\{\beta^{-1}\}$ for the case of $N_T = 3$ using a 16-QAM based codebook. The criss-crossed lines on the figure show where the boundaries of the nearest neighbor regions of the codebook vectors intersect with the plane $\beta^{-1}\mathbf{w}^{\text{opt}}$, where we show the case for an arbitrary \mathbf{w}^{opt} . Seen from a different perspective, the plane is partitioned into regions, such that all values of β^{-1} in a particular region correspond to the same codeword; which by definition is the codeword that minimizes $\|\mathbf{w}^{\text{opt}} - \beta\mathbf{v}\|^2$ for those values β^{-1} .

The search algorithm can be viewed as picking a value of β^{-1} within each region, and for each region calculating the corresponding codeword via nearest neighbor decoding of the point $\beta^{-1}\mathbf{w}^{\text{opt}}$ (For the details on how these boundaries, internal points, and nearest neighbor codewords are calculated, see [15, 18]). It was shown in [15] that this set of codewords contains the codeword which minimizes (9).

It was shown in [18] that since square QAM constellations exhibit 4-fold rotational symmetry, only values of β^{-1} in the first quadrant need to be examined to guarantee that the optimal QAM codeword is found, reducing N_C by approximately a factor of four. Moreover, it was shown in [18, Theorem 2], that knowledge of the range of values of the elements in \mathbf{w}^{opt} can be used to restrict the search over β^{-1} to a finite region. These two factors reduce the search over β^{-1} to that of the shaded region in Figure 1(a). These improvements result in the worst case complexity of the overall algorithm to be $N_C \leq \frac{1}{2}N_T(2N_T - 1)[(\sqrt{M} - 1)^2 - 1] + 1$.

For the case of rectangular 8-QAM, the square QAM algorithm is applied similarly, with the exception that it has only 2-fold rotational symmetry, and the resulting worst-case complexity for this case is $N_C \leq 6N_T^2 - 4N_T + 1$.

Clearly, these methods provide a huge complexity improvement over an exhaustive search, which would require examination of $N > M^{N_T-1}$ codewords.

2) *Suboptimal QAM-SVQ*: For suboptimal QAM-SVQ, the proposed search algorithm from [18, Section VI.B] performs a search which when viewed in terms of (9) corresponds to the case where \mathcal{R} is reduced to a set of L lines emanating radially from the origin, given by $\left\{ \alpha e^{j\theta} \mid \alpha > 0, \theta \in \left\{ 1, e^{\frac{j\pi}{rL}}, \dots, e^{\frac{j2\pi(L-1)}{rL}} \right\} \right\}$, where r is the number of rotational symmetries in the constellation. This is demonstrated in Figure 2(a), where the lines are shown in bold. The search algorithm in this case divides the lines into segments defined by the nearest neighbor boundaries, and calculates corresponding codewords for each segment. This is clearly a lower complexity search compared with

QAM-SVQ described above, since line segments are easier to calculate than two-dimensional region boundaries. The suboptimality comes about since it is clearly not guaranteed that the set of lines pass through all of the regions, therefore some codewords will be missed.

Simulations indicate that the number of lines L , required for near-optimal performance is small; $L = 4$ is sufficient for square constellations where $M \leq 64$ and $N_T \leq 10$. The overall complexity is dominated by the number of phases and the complexity of each ‘line-search’, which is dominated by a sorting operation and therefore the complexity of the algorithm turns out to be $O(N_T \log N_T)$ per phase estimate.

The upper bound on the number of codewords examined by the suboptimal algorithm is only $N_C \leq L(2N_T(\sqrt{M}/2 - 1) + 1)$ for square QAM [18]. For 8-QAM it is $N_C \leq L(2N_T + 1)$, although the number of lines are doubled since 8-QAM requires a search over the right-hand half plane.

3) *Scalar QAM-SVQ*: This algorithm has negligible complexity, at the expense of some performance loss, by performing a simple scalar Euclidean quantization of the codeword. The algorithm works by choosing a single value of β , and then performing a scalar quantization, equivalent to quantizing each element of \mathbf{w}^{opt} independently, thus dropping the second minimization in (9).

We considered several ad hoc methods for choosing the value of β based on knowledge of the constellation and of \mathbf{w}^{opt} . We found that the performance of the methods was very similar, even for small antennas and feedback bits. We therefore suggest using a simple technique by setting β^{-1} such that the average magnitude of the codeword and ideal beamforming vector are matched, *i.e.*

$$\beta = \sqrt{\frac{N}{\sum_{\mathbf{v} \in \mathcal{C}} [\|\mathbf{v}\|^2]}}.$$

We will confirm via simulation in Section VII, that scalar quantization is inferior to the closest-in-angle methods. However, the performance of scalar quantization approaches angular quantization as the number of feedback bits per antenna increases.

4) *PSK-SVQ*: In the case of PSK the cost function (9) has the form

$$\mathbf{w} = \arg \min_{\mathbf{v} \in \mathcal{C}} \min_{\theta \in [0, 2\pi)} \left\| \mathbf{w}^{\text{opt}} - e^{j\theta} \mathbf{v} \right\|^2.$$

This reduction of the search space from $\mathcal{R} = \mathbb{C}$ to $\mathcal{R} = e^{j[0, \pi/2)}$ is due to the constellation having a constant envelope, and therefore the magnitude of β does not affect the corresponding best codeword estimate. For each phase θ a corresponding best codeword estimate exists, and therefore the interval $[0, 2\pi)$ can be partitioned into intervals which correspond to obtaining the same codeword estimate. This reduced search is shown in Figure 1(b) for a quantization of an arbitrary vector \mathbf{w}^{opt} using 8-ary PSK for $N_T = 3$. Note that the search over θ is further reduced to the emboldened arc $[0, \pi/2M)$, due to the rotational symmetry of PSK constellations.

The algorithm in [16,17] effectively calculates the intersection of the arc $[0, \frac{2\pi}{M})$ with the lines on the plane indicating the nearest neighbor boundaries. The algorithm then sorts the boundary points in order of phase which allows the codeword estimates and corresponding angular metric to be updated in a recursive manner.

The complexity of the algorithm is dominated by the sorting operation of N_T boundary points; a computation which has complexity $O(N_T \log N_T)$, and results in $N_C = N_T$.

5) *Scalar PSK-SVQ*: In this scheme the optimal unquantized MRT beamforming vector is rotated so that $w_0^{\text{opt}} = 1$. Then \mathbf{w}_0 is chosen according to the scalar quantization

$$\mathbf{w} = \arg \min_{\mathbf{v} \in \mathcal{C}} \|\mathbf{w}^{\text{opt}} - \mathbf{v}\|^2.$$

VI. PERFORMANCE OF CODEBOOKS USING PROPOSED SEARCH ALGORITHMS

In this section we examine the performance of the new codebooks when used with the proposed codebook search algorithms. We first show that the full diversity order is maintained for the proposed codebook strategies, as it was proved for the exhaustive search in Section IV-A. We also calculate the average effective gain and an upper bound on the capacity of the scalar-PSK technique which provides a lower bound on the performance of the optimal-PSK algorithm.

A. Diversity Order with Proposed Codebook Search Algorithms

Although Lemma 1 covers an exhaustive search approach, it does not guarantee that the use of suboptimal QAM-SVQ or the optimal SVQ approaches for $N_R > 2$ achieves the full diversity order. This is because even though the vector maximizing the SNR may be in the codebook, it may not be examined and chosen by the suboptimal schemes. To cover these scenarios, we first provide the following lemma.

Lemma 2: For an $N_T \times N_R$ wireless system employing a BPSK beamforming codebook, antenna selection at the receiver and codebook search

$$\mathbf{w} = \arg \min_{\mathbf{v} \in \mathcal{C}} \min_{\beta=1,j} \|\mathbf{w}^{\text{opt}} - \beta \mathbf{v}\|^2, \quad (10)$$

the diversity order is $D = N_T N_R$.

Proof: We start by considering a system employing unquantized EGT with antenna selection at the receiver. For such a system the beamforming vector \mathbf{w} has the restriction that $|w_t| = 1/\sqrt{N_T}$, $\forall t = 1, \dots, N_T$, and the combining vector \mathbf{z} is restricted to be one of the rows of the $N_R \times N_R$ identity matrix. The effective channel gain of the ideal unquantized system with EGT and receiver antenna selection (SEL) for a particular \mathbf{H} is

$$\Gamma_{\text{EGT,SEL}}(\mathbf{H}) = \max_{1 \leq r \leq N_R} \frac{1}{N_T} \left(\sum_{t=1}^{N_T} |h_{r,t}| \right)^2 \quad (11)$$

It was shown in [4] that this system achieves a diversity order of $N_T N_R$.

We now show that for the BPSK-based beamforming codebook and using $\mathcal{R} = \{1, j\}$ the SNR is no worse than a quarter of the SNR achieved by using the EGT-SEL system with identical E_x/N_0 reduced by a constant factor. From this it follows that the diversity order is no worse than that for EGT-SEL, and is hence equal to it. The codebook search scheme given in (10) effectively chooses one of the beamforming vectors

$$\mathbf{w}^{(1)} = \frac{1}{\sqrt{N_T}} \text{sgn} \{ \text{Re} \{ h_{r_0,1} \} \} \text{sgn} \{ \text{Re} \{ \mathbf{h}_{r_0} \} \} \quad \text{or} \quad \mathbf{w}^{(2)} = \frac{1}{\sqrt{N_T}} \text{sgn} \{ \text{Im} \{ h_{r_0,1} \} \} \text{sgn} \{ \text{Im} \{ \mathbf{h}_{r_0} \} \} \quad (12)$$

depending on which one gives the greater effective channel gain, where r_0 is the maximizing r in (11), and \mathbf{h}_r is the r th row of \mathbf{H} . The vectors $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ are derived from the cases where $\beta = 1$ and $\beta = j$ in (10) respectively. We also define \mathbf{e}_{r_0} as the r_0 th row of the $N_R \times N_R$ identity matrix.

Using $\mathbf{w} = \mathbf{w}^{(1)}$ and the antenna selection vector $\mathbf{z} = \mathbf{e}_{r_0}$, the effective channel gain is

$$\Gamma_{\text{BPSK,SEL}}^{(1)}(\mathbf{H}) = \frac{1}{N_T} \left| \sum_{t=1}^{N_T} h_{r_0,t} \text{sgn} \{ \text{Re} \{ h_{r_0,t} \} \} \right|^2 \geq \frac{1}{N_T} \left(\sum_{t=1}^{N_T} |\text{Re} \{ h_{r_0,t} \}| \right)^2$$

Similarly, when using $\mathbf{w} = \mathbf{w}^{(2)}$,

$$\Gamma_{\text{BPSK,SEL}}^{(2)}(\mathbf{H}) \geq \frac{1}{N_T} \left(\sum_{t=1}^{N_T} |\text{Im} \{ h_{r_0,t} \}| \right)^2.$$

Hence, when choosing the best of the two vectors, the effective channel gain is,

$$\begin{aligned} \Gamma_{\text{BPSK,SEL}}(\mathbf{H}) &\geq \max \left\{ \Gamma_{\text{BPSK,SEL}}^{(1)}(\mathbf{H}), \Gamma_{\text{BPSK,SEL}}^{(2)}(\mathbf{H}) \right\} \\ &\geq \frac{1}{2} \left(\Gamma_{\text{BPSK,SEL}}^{(1)}(\mathbf{H}) + \Gamma_{\text{BPSK,SEL}}^{(2)}(\mathbf{H}) \right) \\ &\geq \frac{1}{4N_T} \left(\sum_{t=1}^{N_T} |\text{Re} \{ h_{r_0,t} \}| + \sum_{t=1}^{N_T} |\text{Im} \{ h_{r_0,t} \}| \right)^2 \\ &\geq \frac{1}{4N_T} \left(\sum_{t=1}^{N_T} |h_{r_0,t}| \right)^2 = \frac{1}{4} \Gamma_{\text{EGT,SEL}}(\mathbf{H}) \end{aligned}$$

where the third inequality follows from the fact that for any $a, b \in \mathbb{R}$, $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ and the fourth inequality is obtained by application of the triangle inequality.

Now, for all \mathbf{H} , we have shown that by choosing the BPSK vectors as in (12), we have $\Gamma_{\text{BPSK,SEL}}(\mathbf{H}) \geq \frac{1}{4} \Gamma_{\text{EGT,SEL}}(\mathbf{H})$. Hence the probability of error is no more than that of a system using EGT-SEL with transmit power $\frac{1}{4} E_x$. Thus the diversity order is no less than that of an EGT-SEL system, which was previously established to be $N_R N_T$. Since it is not possible to be greater than $N_R N_T$, the BPSK-based beamforming codebook diversity must equal $N_R N_T$. \blacksquare

The scalar quantization scheme proposed in the proof of Lemma 2 in (12) performs a search over $\beta \in \mathcal{R} = \{1, j\}$ so as to minimize $\|\mathbf{w}^{\text{opt}} - \beta \mathbf{v}\|^2$. As discussed in Section V, the QAM-SVQ and PSK-SVQ algorithms

are equivalent to performing a search over all $\beta \in \mathcal{R} = \mathbb{C}$, and therefore the effective channel gain is never less than that of the scheme of Lemma 1. For suboptimal QAM-SVQ, the search region is effectively $\mathcal{R} = \left\{ \alpha e^{\frac{j\pi\ell}{2L}} \mid \alpha \in [0, \infty), \ell = 0, \dots, 3L - 1 \right\}$ if the rotational symmetry of the constellation is taken into account; which clearly includes $\{1, j\}$. Hence the effective channel gain of suboptimal QAM-SVQ, and the QAM-SVQ and PSK-SVQ schemes when $N_R > 2$, is never less than that of the scheme used in Lemma 1. Therefore the full diversity order is achieved when using these codebook search schemes.

B. Performance of scalar PSK-SVQ

In the case of an $N_T \times 1$ system using scalar PSK-SVQ it is possible to obtain explicit expressions for the average effective channel gain, which can be used to obtain a capacity upper bound.

Lemma 3: For an $N_T \times 1$ wireless system employing scalar PSK-SVQ of an M -ary PSK-based beamforming codebook, the average effective channel gain is

$$\Gamma_{\text{Scalar } M\text{-PSK,ave}} = \Omega^2 \Gamma_{\text{EGT,ave}} + (1 - \Omega^2) + \frac{\pi(N_T - 1)}{2N_T} [\Omega - \Omega^2] \quad (13)$$

where $\Omega \triangleq \text{sinc}(M^{-1})$ and $\Gamma_{\text{EGT,ave}}$ is the average effective channel gain for unquantized EGT.

Proof: For a particular channel $\mathbf{h} \in \mathbb{C}^{1 \times N_T}$, the effective channel gain is determined by the difference between the channel phase and the phase of the beamforming vector chosen by scalar quantization. We first apply a rotation to the channel vector so that h_1 has zero imaginary component. It follows that the difference in phase between w_t and h_t^* for $t > 1$ is at most π/M . Hence, defining $\alpha_t = \frac{M}{2\pi}(\arg h_t^* - \arg w_t)$ the effective channel gain of a scalar M -ary PSK system for a particular \mathbf{h} is given by

$$\Gamma_{\text{Scalar } M\text{-PSK}}(\mathbf{h}) = \frac{1}{N_T} \left| \sum_{t=1}^{N_T} |h_t| e^{\frac{j2\pi\alpha_t}{M}} \right|^2$$

where $\alpha_1 = 0$ and $\alpha_t \in [-0.5, 0.5]$ for $t = 2, \dots, N_T$.

We denote $\mathbf{a} = [a_1 \dots a_{N_T}]$ as the vector of amplitudes of \mathbf{h} , that is $a_t \triangleq |h_t|$. We now calculate the average gain over all channels \mathbf{h} which have the same \mathbf{a} . First note that the phase of each element of \mathbf{h} is uniformly distributed over the interval $[0, 2\pi)$ and independent of the amplitude a_t . Therefore α_t is an i.i.d. uniformly distributed variable over the interval $[-0.5, 0.5]$ which gives

$$E_{\mathbf{h}|\mathbf{a}}[\Gamma_{\text{Scalar } M\text{-PSK}}(\mathbf{h})] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \dots \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N_T} \left| \sum_{t=1}^{N_T} |h_t| e^{\frac{j2\pi\alpha_t}{M}} \right|^2 d\alpha_2 \dots d\alpha_{N_T}.$$

We now use the following properties. For $t \neq t'$ and $t, t' \neq 1$ we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\frac{j2\pi\alpha_t}{M}} e^{-\frac{j2\pi\alpha_{t'}}{M}} d\alpha_t d\alpha_{t'} = \Omega$$

since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos\left(\frac{2\pi\alpha}{M}\right) d\alpha = \Omega \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\left(\frac{2\pi\alpha}{M}\right) d\alpha = 0.$$

Hence,

$$\begin{aligned} E_{\mathbf{h}|\mathbf{a}}[\Gamma_{\text{Scalar } M\text{-PSK}}(\mathbf{h})] &= \frac{1}{N_T} \left[\sum_{t=1}^{N_T} |h_t|^2 + 2|h_1|\Omega \sum_{t=2}^{N_T} |h_t| + \sum_{t=2}^{N_T} \sum_{\substack{t'=2 \\ t' \neq t}}^{N_T} |h_t||h_{t'}|\Omega \right] \\ &= \frac{1}{N_T} \left[\Omega^2 \left(\sum_{t=1}^{N_T} |h_t| \right)^2 + (1 - \Omega^2) \|\mathbf{h}\|^2 + 2|h_1| \sum_{t=2}^{N_T} |h_t|[\Omega - \Omega^2] \right] \end{aligned}$$

and hence by averaging over \mathbf{a} we obtain,

$$E[\Gamma_{\text{Scalar } M\text{-PSK}}(\mathbf{h})] = \Omega^2 \Gamma_{\text{EGT,ave}} + (1 - \Omega^2) \frac{E[\|\mathbf{h}\|^2]}{N_T} + \frac{2}{N_T} E[|h_1|] \sum_{t=2}^{N_T} E[|h_t|][\Omega - \Omega^2].$$

where we recall that $\Gamma_{\text{EGT,ave}}$ is the average effective channel gain for the case of perfect feedback. Noting that $E[\|\mathbf{h}\|^2] = N_T$ and $E[|h_t|] = \sqrt{\pi}/2$ for all t completes the proof. \blacksquare

As expected, since $\Omega = \text{sinc}(M^{-1}) \rightarrow 1$ as $M \rightarrow \infty$ the average effective channel gain of the scalar quantization system approaches that of the ideal unquantized EGT system as $M \rightarrow \infty$. By noting that $\log(1 + \Gamma_{\text{Scalar } M\text{-PSK}}(\mathbf{h}))$ is concave and applying Jensen's inequality gives the following corollary.

Corollary 1: The capacity C of an $N_T \times 1$ wireless system employing scalar PSK-SVQ of an M -ary PSK-based beamforming codebook with antenna selection at the receiver is

$$C \leq \log_2 \left(1 + \frac{E_x}{N_0} \Gamma_{\text{Scalar } M\text{-PSK,ave}} \right)$$

where $\Gamma_{\text{Scalar } M\text{-PSK,ave}}$ is given by (13).

VII. SIMULATION RESULTS

In this section we perform Monte Carlo simulations to examine the average effective channel gain, bit error rate and the number of codeword metric calculations required for the QAM-based codebooks using the proposed optimal, suboptimal and scalar quantization codebook search algorithms given in Section V.

We compare the QAM and PSK based codebooks with codebooks of the same size, namely RVQ [8] and with the numerically determined codebooks in [3, 19, 20], which are available for small N_T and codebook size N , and can be considered near-optimal. For RVQ, the codewords are each chosen independently and isotropically from the feasible set of MRT or EGT beamforming vectors. We also average over the RVQ codebooks. The receiver employs MRC.

For the suboptimal QAM-SVQ algorithm of Section V-C2 we use $L = 4$ different line searches: for square QAM constellations the line searches are separated by $\frac{\pi}{2L}$ radians, whereas for the rectangular 8-QAM constellation the line searches are separated by $\frac{\pi}{L}$.

Experiment 1: In Figure 3, we show the average effective channel gain $\Gamma_{\text{ave}} \triangleq E[\Gamma(\mathbf{H})]$, (where the expectation is over \mathbf{H}) as a function of $\log_2 M$, the number of bits for the respective QAM constellation. Plots are shown for a system with $N_R = 2$ receive antennas and $N_T = 2, 4, 8$ transmit antennas. As an upper bound, the performance of unquantized MRT is also plotted. For the codebook search, the three QAM algorithms in Section V are used for the QAM-based codebooks, while an exhaustive search is employed for the unstructured codebooks. Note that no plots are shown for RVQ systems for $N_T = 8$ when using $\log_2 M = 3, 4$ as in excess of 8×10^6 codewords are required and this is computationally infeasible.

We see that the performance of the QAM codebooks using the optimal and suboptimal quantization algorithms are nearly identical, and the performance is comparable to that obtained by the unstructured codebooks (both RVQ and numerically optimized). Also, the performance approaches unquantized MRT as $M \rightarrow \infty$, as explained in Section III-A. Note that the performance of the scalar quantization technique for the QAM codebooks is clearly inferior to the other schemes but does approach the performance of angular quantization as the number of feedback bits per antenna increases.

The figure also indicates that from a practical point of view it would be preferable to choose the QAM codebooks with B bits over an RVQ codebook of the same size, even though RVQ is asymptotically optimal for large N, N_T [9, 10]. This is because we see that for small N_T the performance of the QAM codebooks is superior to RVQ (e.g. for $\log_2 M = 1$, QAM collapses to BPSK which is optimal for the case where $N_T = N$ as the codebook vectors are orthogonal), while for $N_T \geq 8$ RVQ is computationally infeasible.

Experiment 2: In Figure 4, we show the average effective channel gain as a function of $\log_2 M$ for PSK codebooks. Plots are shown for a system with $N_R = 1$ receive antennas and $N_T = 2, 4, 8$ transmit antennas. We plot the performance for both optimal and scalar PSK-SVQ, where the scalar PSK-SVQ curves coincide with (13). We compare with RVQ codebooks where the codewords are each chosen independently and isotropically, and average over the RVQ codebooks. We also compare with numerically derived codebooks obtained using the Lloyd-Max algorithm based codebooks [6]. The performance of unquantized EGT is also plotted.

We see that the performance of the PSK codebooks is better than that of RVQ, and choosing $\log_2 M \geq 3$ is sufficient to achieve performance close to perfect feedback. The average effective channel gain of scalar PSK-SVQ matches the analytical result given by Lemma 3, and performs very closely to PSK-SVQ for $\log_2 M \geq 3$.

Experiment 3: In Figure 5, we show the bit error rate of BPSK symbol detection for the proposed QAM codebooks

as a function of E_x/N_0 . Plots are shown for $M = 4, 8, 16$ and the unquantized MRT upper bound. We see that the performance approaches that of MRT as M increases and that for $M = 16$, the curve is within 0.2dB of the optimal for $N_T = 2, 4, 8$. For $N_T = 8$, the performance of the QAM schemes is within 1.8, 0.9, 0.13 dB, for $M = 4, 8, 16$ respectively. Thus, again we see the asymptotic convergence to the ideal performance.

Experiment 4: In Figure 6 we examine the cumulative distribution function (c.d.f.) of $\cos^2 \theta(\mathbf{w}, \mathbf{w}^{\text{opt}})$ representing the angular distortion between the quantized and ideal beamforming vectors. Plots are shown for an $N_T = 8, N_R = 1$ system using 8-QAM codebooks along with RVQ. The codebooks are all of size $N = 2^{23}$. For RVQ, the outage probability is given in [28, Lemma 1]. We see that in this case RVQ is superior. However, in Figure 7 we see that this only results in an approximately 0.16dB difference between QAM-SVQ and RVQ in terms of the outage probability of the effective channel gain $\Pr\{\Gamma(\mathbf{h}) < \gamma\}$. This can be explained by first noting that $\cos^2 \theta(\mathbf{w}, \mathbf{h})$ is between 0.25 and 1 if there is at least one feedback bit per antenna (this can be proved using the method in the proof of Lemma 2), whereas $\|\mathbf{h}\|^2$ has infinite range as it is determined by a chi-square distribution. Hence the order of magnitude of the outage probability is approximated to a first order by the c.d.f. of $\|\mathbf{h}\|^2$.

Experiment 5: In Figure 8, we examine the number of codeword metric calculations N_C required to obtain a given average effective channel gain Γ_{ave} . Plots are shown for $N_T = 4, 8$ with $N_R = 2$, for both QAM codebooks using the optimal and suboptimal algorithms and also for RVQ using an exhaustive search. As N_C is variable for the suboptimal algorithm we plot the average complexity. The unquantized MRT upper bound is also shown and the number of feedback bits is also given. For the QAM constellations, this corresponds to $M = 4, 8, 16, 64$. Note that we do not consider QPSK for the suboptimal algorithm as the faster PSK-SVQ algorithm can be used instead. Also, note that the curves for our proposed QAM codebooks are not smooth due to the various efficiencies we are able to achieve for the different constellations as discussed in Sections V-C1 and V-C2. We also note that suboptimal vector quantization algorithms could be applied however these require full storage of the codebook [5, 14].

For the case $N_T = 4$, note that when $B = 14$ feedback bits are employed we are within 0.1 dB of the RVQ performance and that 16384 metric calculations are required for RVQ, while only 113 are required for the optimal QAM algorithm and on average 28 are required using the suboptimal algorithm (with $M = 16$). Clearly, even for moderate numbers of antennas, there are huge complexity reductions to be obtained by using QAM codebooks and the proposed algorithms instead of exhaustive search techniques.

Similarly, for $N_T = 8$, $B = 14$ to be within 1.2 dB of the optimal gain, 16384 metric calculations are required for RVQ, while only 8 are required by using $M = 4$ QAM codebooks. Also note that near optimal performance is obtained using 64-QAM for both $N_T = 4$ and 8, where the respective codebook sizes are $N \approx 4 \times 10^6$ ($B = 22$

bits) and $N \approx 7 \times 10^{13}$ ($B = 46$ bits) respectively. Using such large codebooks is simply infeasible for exhaustive search based schemes.

We see that the complexity of the suboptimal QAM-SVQ algorithm is a factor of ~ 10 less than that of QAM-SVQ for $N_T = 4$, and a factor of ~ 50 less for $N_T = 8$. Since suboptimal QAM-SVQ performs nearly as well as QAM-SVQ, it is an even more attractive choice for implementation. Finally, we again note that the performance of the scalar quantization technique for the QAM codebooks is clearly inferior to the other schemes but does approach the performance of angular quantization as the number of feedback bits per antenna increases.

VIII. CONCLUSION

In this paper we have proposed schemes for codebook construction and searching which deliver near-ideal performance for limited-rate feedback MIMO systems. We have shown that structured codebooks based on QAM and PSK sequences perform comparably to the unstructured random and numerically derived codebooks, in terms of both average SNR, bit error rate and outage probability. Taking advantage of the structure of QAM and PSK sequences, we show that these codebooks can be searched in an optimal manner with complexity orders of magnitude smaller than an exhaustive search. Furthermore, by using the suboptimal algorithm of [18], very fine quantization can be achieved with very low computational expense. We have shown analytically that the proposed codebooks and algorithm combinations obtain the full-diversity order, and have provided a tight lower bound on the performance of the PSK codebooks.

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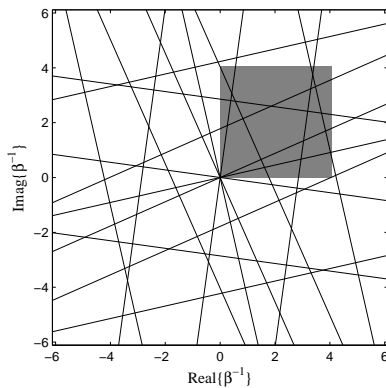
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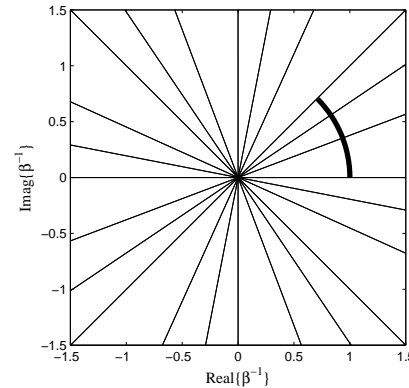
1	begin
2	$\mathbf{w}^{\text{opt}} :=$ dominant right singular vector of \mathbf{H} ;
3	if QAM and $M > 4$;
4	$\mathbf{w} :=$ QAM-SVQ(\mathbf{w}^{opt}) ;
5	Rotate-To-First-Quadrant(\mathbf{w})
10	return bits := QAM-to-Bits(\mathbf{w});
6	else if PSK or (QAM and $M = 4$)
7	$\mathbf{w} :=$ PSK-SVQ(\mathbf{w}^{opt}) ;
8	Rotate-To-Real-Axis(\mathbf{w})
10	return bits := PSK-to-Bits(\mathbf{w});
9	end if ;

TABLE I

ALGORITHM FOR FAST QAM CODEBOOK SEARCH

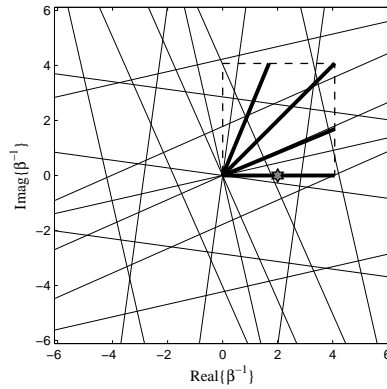


(a) QAM-SVQ



(b) PSK-SVQ

Fig. 1. Visual representation of optimal reduced search algorithm for $N_T = 3$ codebook search for (a) 16-ary square QAM and for (b) 8-ary PSK codebooks. Each polygonal region on the plot corresponds to the values of $\beta \in \mathbb{C}$ which produce the same codeword estimate. Due to properties of the constellation, the optimal search is reduced to (a) the shaded region for QAM-SVQ; and (b) the emboldened arc for PSK-SVQ.



(a) Suboptimal and Scalar QAM-SVQ

Fig. 2. Visual representation of suboptimal and scalar QAM-SVQ reduced search algorithms for $N_T = 3$ codebook search for a 16-ary square QAM codebook. Suboptimal approaches reduce the search region to the emboldened lines indicating multiple ‘line-searches’ for suboptimal QAM-SVQ; and a single point (denoted here as a star) for scalar QAM-SVQ.

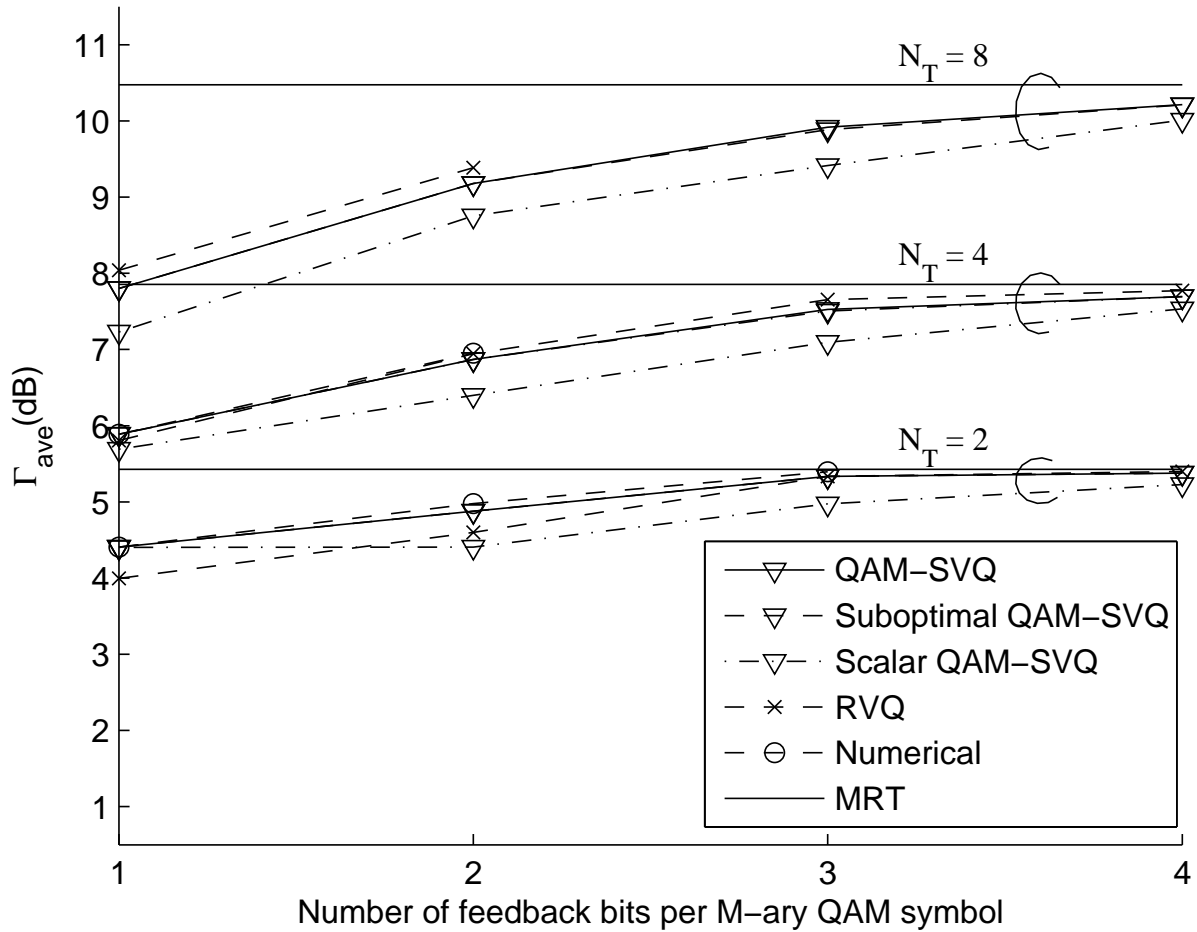


Fig. 3. Plot of average effective channel gain Γ_{ave} for QAM codebooks as a function of the number of bits per QAM symbol. Plots are shown for $N_T = 2, 4, 8$ transmit antennas with $N_R = 1$ receive antennas. The effective channel gains for RVQ and some numerically found codebooks for quantized maximum ratio transmission are also shown.

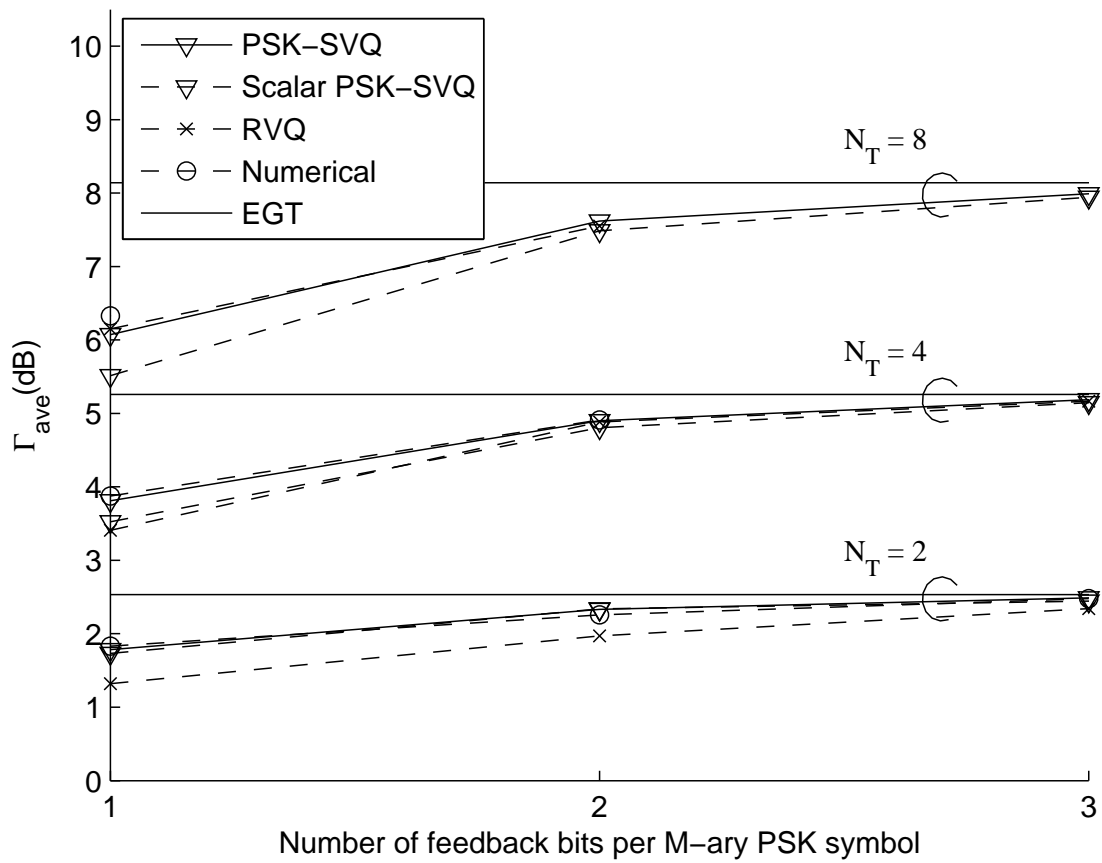


Fig. 4. Plot of average effective channel gain Γ_{ave} for PSK, RVQ and Lloyd-Max based codebooks for quantized EGT as a function of number of bits per PSK symbols. Plots are shown for $N_T = 2, 4, 8$ transmit antennas with $N_R = 1$ receive antennas.

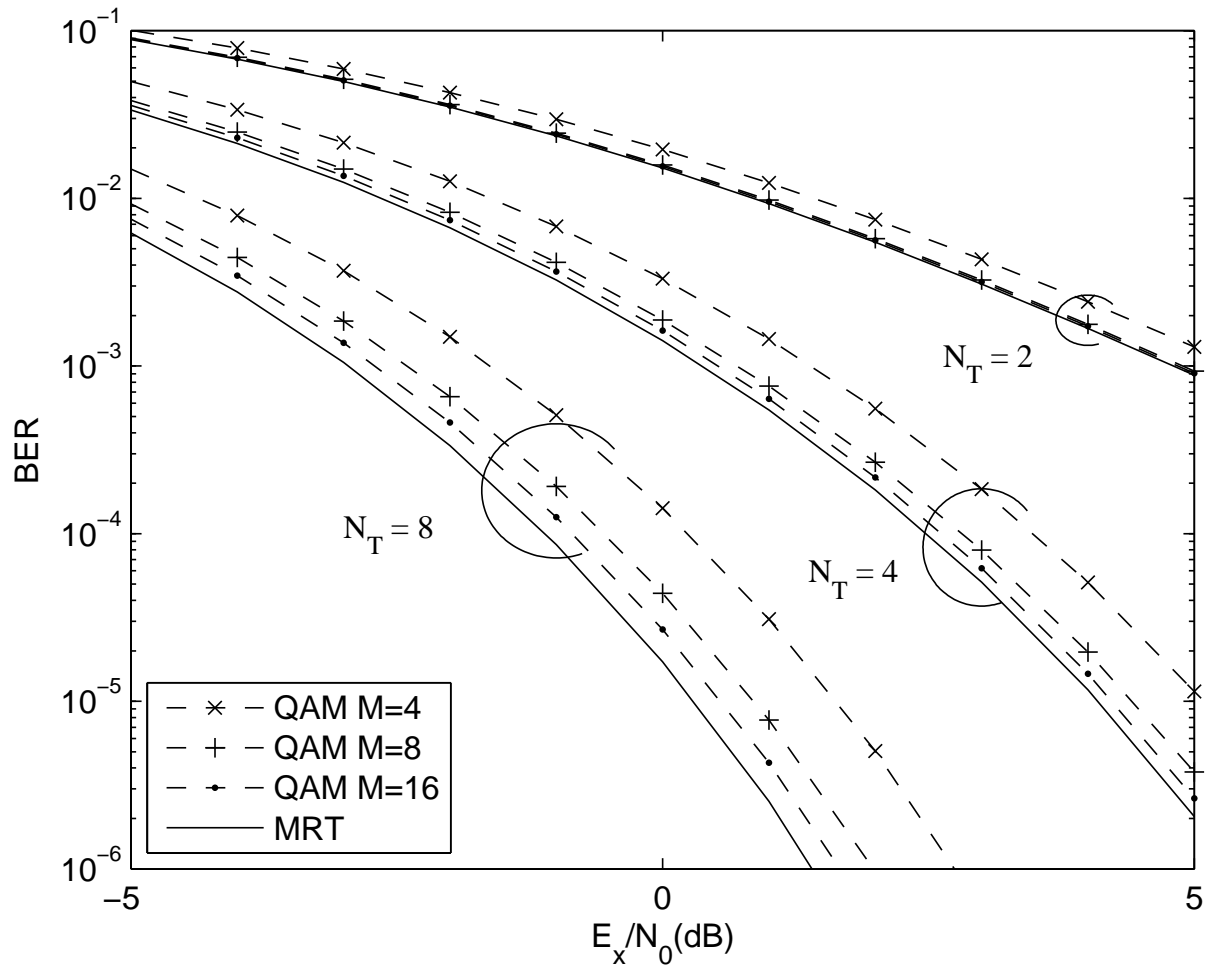


Fig. 5. Plot of BPSK BER using QAM beamforming codebooks as a function of E_s/N_0 . Plots are shown for $N_T = 2, 4, 8$ transmit antennas with $N_R = 2$ receive antennas.

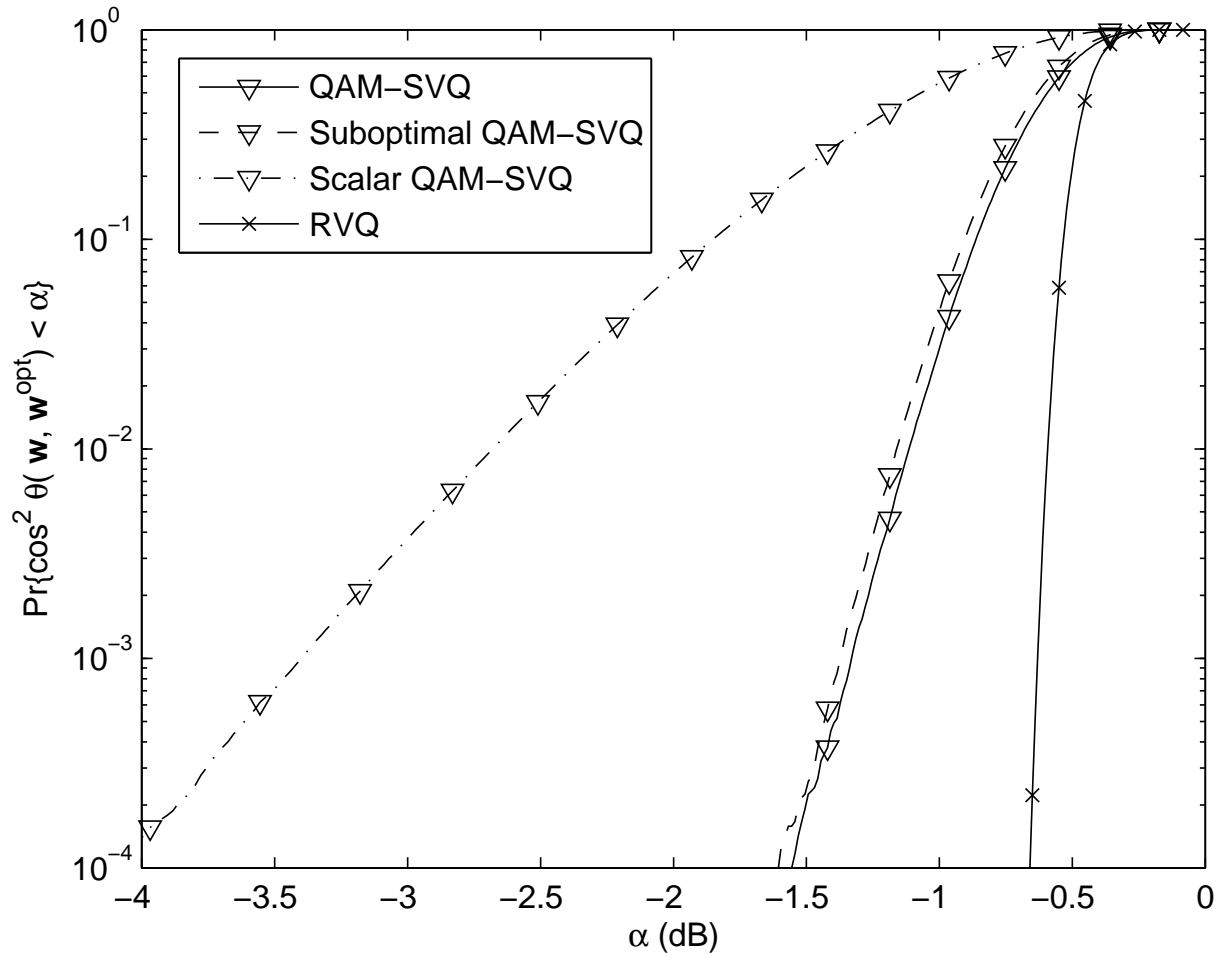


Fig. 6. Plots of the c.d.f. $\Pr(\cos^2 \theta(\mathbf{w}, \mathbf{w}^{\text{opt}}) < \alpha)$ for an 8-ary QAM codebook using the proposed search strategies of Section V compared to RVQ. Plots are shown for $N_T = 8$ transmit antennas with $N_R = 1$ receive antenna.

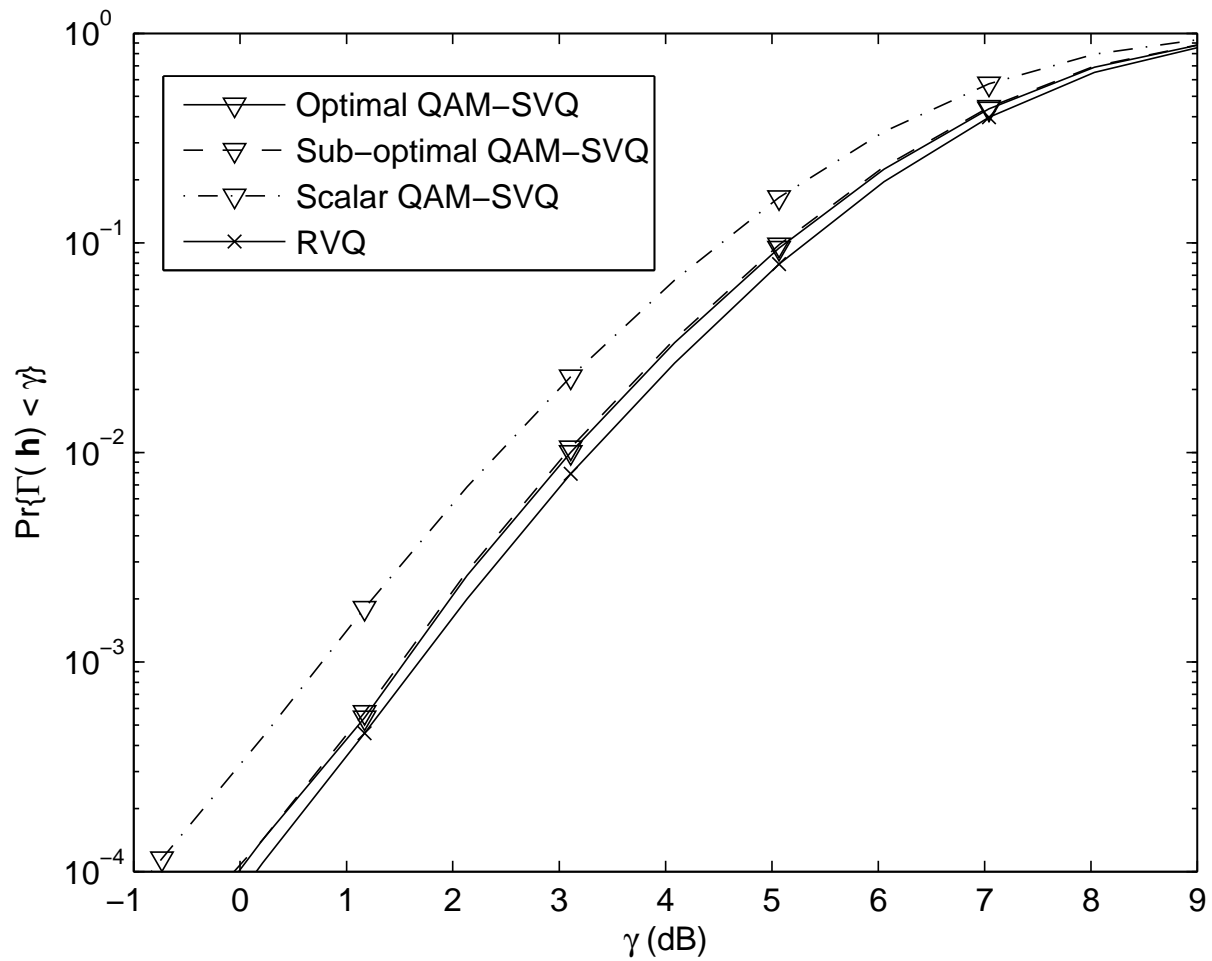


Fig. 7. Plots of the resulting outage probability $\Pr(\Gamma(\mathbf{h}) < \gamma)$ for the same scenario as Figure 6.

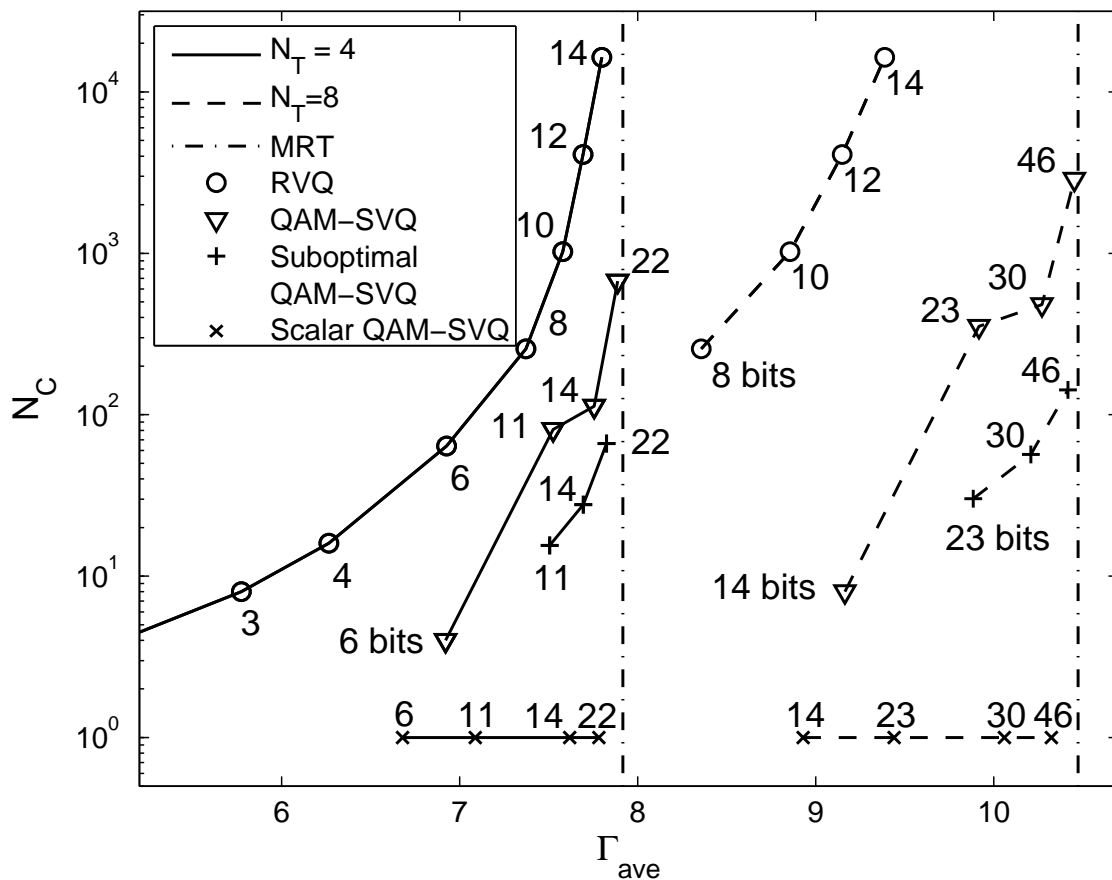


Fig. 8. Plot of number of metric calculations N_C for QAM and RVQ codebooks as a function of the average effective channel gain Γ_{ave} . Plots are shown for $N_T = 4, 8$ transmit antennas with $N_R = 2$ receive antennas.